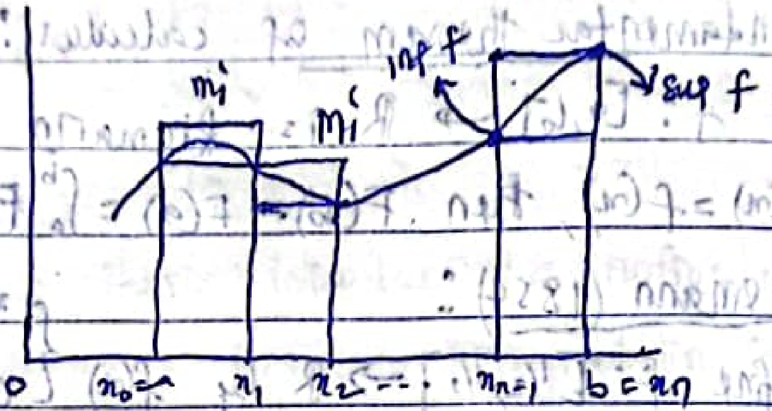
MATH 408: Measure and Integration

Riemann integral: $f : [a,b] \longrightarrow \mathbb{R}$



$P = \{x_0 = a, x_1, x_2, \ldots, x_{n-1}, x_n = b\}$ $x_i < x_{i-1}$ $(0 \le i \le n-1)$

Let $m_i = \inf\{f(x) \mid x_{i-1} \le x \le x_i\}$

$M_i = \sup\{f(x) \mid x_{i-1} \le x \le x_i\}$

$U(P,f) = \sum_{i=1}^{n} M_i (x_i - x_{i-1})$ —— Upper sum

$L(P,f) = \sum_{i=1}^{n} m_i (x_i - x_{i-1})$ —— Lower sum

$m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$

If P_2 is a refinement of P_1, then

$L(P,f) \le L(P_2,f) \le U(P_2,f) \le U(P,f)$

Lower integral of f: $\underline{\int_a^b} f(x)\,dx = \sup_P \{L(P,f)\}$

Upper integral of f: $\overline{\int_a^b} f(x)\,dx = \inf_P \{U(P,f)\}$

We say f is Riemann integrable if

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$$

$f(3) = f(3-1) = f(2) = f(2-1) = f(1) = f(1-1) = f(0) = 0$

In this case we call the common value Riemann integral of f denoted by $\int_a^b f(x)\,dx$.

Riemann integral satisfies the following:

Fundamental theorem of calculus:

If $f:[a,b] \to \mathbb{R}$ is Riemann integrable and $F'(x) = f(x)$, then $F(b) - F(a) = \int_a^b f(x)\,dx$.

Riemann (1854):

Define $f:[\frac{1}{2}, \frac{1}{2}] \to \mathbb{R}$ by $f(x) \begin{cases} x & \text{if } -\frac{1}{2} \leq x \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$

And extend F to $\mathbb{R}$ such that $f(x) = f(x+1)$.

Define also $R: \mathbb{R} \to \mathbb{R}$ by $R(x) = \sum_{n=1}^{\infty} \frac{f(nx)}{n^2}$, $x \in \mathbb{R}$.

Then R is Riemann integrable.

Hermann Hankel (1871):

Define $G(x) = \int_0^x R(t)\,dt$, Then $G(x)$ is continuous everywhere but not differentiable at an infinite set of points.

Karl Weierstrass (1872)

constructed a function that is everywhere continuous but nowhere differentiable.

Therefore, we can no longer claim that integration (in the Riemann sense) is the reverse of differentiation. That is, It is not the case that F and $F \in R[a,b]$ are such that $F(b) - F(a) = \int_a^b f(x)\,dx \Leftrightarrow F(x)$ is differentiable and $F'(x) = f(x)$.

4.0 Integration

4.1 Lebesgue integration of non-negative simple fu

4.2 Integral of non-negative measurable functions

4.3 Integrable functions

4.4 Lebesgue integral and its relation with Riemann integral.

4.5

5.0 Fundamental theorem of calculus for Lebesgue integrals

5.1 Absolutely continuous functions

5.2 Differentiability of monotone functions.

5.3 Fundamental theorem of calculus and its applicatio

0.0 Prologue.

0.1 Extended real line

Let $\mathbb{R}$ be the set of real numbers. Then the extended real line is the set $\mathbb{R}^* = \mathbb{R} \cup \{+\infty, -\infty\}$, $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}^* = [-\infty, +\infty]$ with the new symbols $-\infty$ and $+\infty$ one defines order and relation and (algebraic) operation on $\mathbb{R}^*$ by:

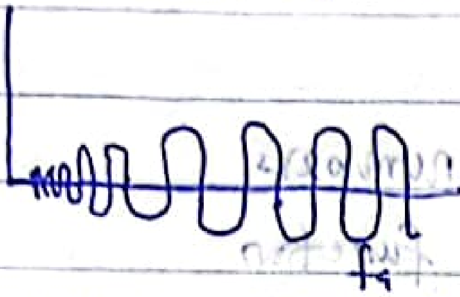
i. $x \in \mathbb{R}$, $-\infty < x < +\infty$

ii. $\forall x \in \mathbb{R}$, $(-\infty) + x = -\infty$ and $+\infty + x = +\infty$

$(+\infty) + (+\infty) = +\infty$ and $(-\infty) + (-\infty) = -\infty$

Vito Voltera (1881): Let $a \in \mathbb{R}$, define $f_a : \mathbb{R} \to \mathbb{R}$

$$f_a(x) = \begin{cases} (x-a)\sin \frac{1}{x-a}, & \text{if } x \neq a \\ 0, & \text{if } x = a \end{cases}$$



Henri Lebesque (1902):

In his Ph.D Thesis introduces a notion of integral which generalizes Riemann integration and take care of the drawback of Riemann integral. This integral goes by the name Lebesgue integral. It is this, his work that lead to the abstract theory of measure and integration.

The goal of this course MATH 408 is to cover basics of measure and integration. The objectives are:

1. Generalization of concept of area and volume
2. Extend the concept of integration to more general settings. Leading to Lebesgue integration for functions on $\mathbb{R}$.

Text book:

An introduction to measure and integration by Inder K. Rana (2002).

Course content.

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4.0 Integration

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5.0 Fundamental theorem of calculus for Lebesgue integrals

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5.3 Fundamental theorem of calculus and its application

0.0 Prologue.

0.1 Extended real line

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i. $x \in \mathbb{R}$, $-\infty < x < +\infty$

ii. $\forall x \in \mathbb{R}$, $(-\infty) + x = -\infty$ and $+\infty + x = +\infty$

$(+\infty) + (+\infty) = +\infty$ and $(-\infty) + (-\infty) = -\infty$

iii. $\forall x \in \mathbb{R}$,

$$\left.\begin{array}{l} x(+\infty) = (+\infty)x = +\infty \\ x(-\infty) = (-\infty)x = -\infty \end{array}\right\} x > 0$$

$$\left.\begin{array}{l} x(+\infty) = (+\infty)x = -\infty \\ x(-\infty) = (-\infty)x = +\infty \end{array}\right\} x < 0$$

$$0(+\infty) = 0(-\infty) = 0$$

$$(\pm\infty)(+\infty) = \pm\infty$$

$$(\pm\infty)(-\infty) = \mp\infty$$

iv. The operations:

$-\infty + (+\infty)$ and $(+\infty) + (-\infty)$ are not defined. Also, $(+\infty) - (+\infty)$ and $(-\infty) - (-\infty)$ are not defined. For a non-empty subset $A \subseteq \mathbb{R}^*$, we write. $\sup A = +\infty$ If $A \cap \mathbb{R}$ is not bounded above, and we write the $\inf A = -\infty$. If $A \cap \mathbb{R}$ is not bounded below. Thus $\sup A$ and $\inf A$ always exist in $\mathbb{R}^*$.

Limits of Sequences in $\mathbb{R}^*$

For any monotonically increasing sequence $\{x_n\}_{n \geq 1}$ in $\mathbb{R}^*$ which is not bounded above we say $\{x_n\}_{n \geq 1}$ is converged to $+\infty$ and write

$$\lim_{n \to \infty} x_n = +\infty.$$

Similarly, If $\{x_n\}_{n\geq 1}$ is a monotonically decreasing sequence in $\mathbb{R}^*$, which is not bounded below, we say $\{x_n\}_{n\geq 1}$ converges to $-\infty$ and $\lim_{n\to\infty} x_n = -\infty$.
Thus, in $\mathbb{R}^*$ every monotone sequence is convergent.

Therefore, in $\mathbb{R}^*$, for any sequence $\{x_n\}_{n\geq 1}$, the sequence $\{\sup_{k\leq i} x_k\}_{i\geq 1}$ and $\{\inf_{k\leq i} x_k\}_{i\geq 1}$ always converges. We then write

$$\lim_{n\to\infty} \sup x_n = \lim_{i\to\infty} \sup_{k\leq i} x_k \quad \text{and} \quad \lim_{n\to\infty} \inf x_n =$$

$\lim_{n\to\infty} \inf x_k$. These limits are respectively called limit superior and limit inferior, of $\{x_n\}_{n\geq 1}$. We say a sequence $\{x_n\}_{n\geq 1}$ is <u>convergent</u> to $x\in\mathbb{R}^*$, If the $\lim_{n\to\infty} \inf x_n = \lim_{n\to\infty} x_n = x$.

<u>0.2 The length function</u>

Let I be the collection of all intervals of $\mathbb{R}$.
If an interval $I \in \mathcal{I}$ is finite with end points a and b. we write $I = I(a,b)$.
For any $a\in\mathbb{R}$, $\phi = (a,a)$, $\mathbb{R} = (-\infty,\infty)$.

<u>Definition</u>

The function $\lambda : \mathcal{I} \to [0,\infty]$ defined by

$$\lambda(I(a,b)) = \begin{cases} |b-a|, & \text{If } a,b\in\mathbb{R} \\ +\infty, & \text{If } a=-\infty \text{ or } b=+\infty. \end{cases}$$

is called the length function on $\mathbb{R}$. The length function l has the following properties:

Lemma 0.1

$l(\emptyset) = 0$, $l(\mathbb{R}) = +\infty$

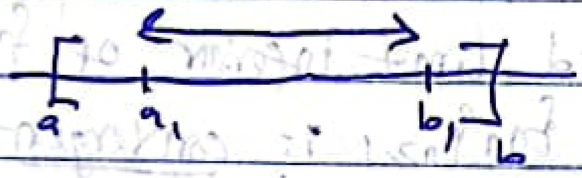
Lemma 0.2 (Monotonicity property)

For all $I, J \in \mathcal{I}$, if $I \subseteq J$, then $l(I) \leq l(J)$

proof

If $J(a,b)$, $I = (a_1, b_1)$

~~then $a < a$~~



then $a \leq a_1 < b_1 \leq b$. But then

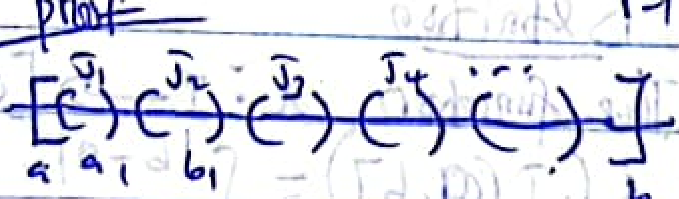
$$l(I) = b_1 - a_1 \leq b - a_1$$
$$\leq b - a$$
$$= l(J) \quad \square$$

Lemma 0.3 (Finite additivity of l)

Let $I \in \mathcal{I}$ be such that $I = \bigcup_{i=1}^{n} J_i$ where $J_i \cap J_j = \emptyset$ for $i \neq j$. Then $l(I) = \sum_{i=1}^{n} l(J_i)$

proof

~~If I is a~~



If I is infinite, then at least one of the J_i's is infinite also. Thus, $l(I) = +\infty = \sum_{i=1}^{n} l(J_i)$.

If I is finite with $I = I(a,b)$, $a,b \in \mathbb{R}$ and $a<b$, then each J_i is finite. also $J_i = J_i(a_i;b_i)$, $b_i \in \mathbb{R}$ and $a_i < b_i$. We may assume (without loss of generality) that $a = a_1 < b_1 = a_2 < b_2 = a_3 < b_3 = \dots = a_n < b_n = b$. Then

$$\lambda(I) = b-a = b_n - a_1 = \sum_{i=1}^{n} b_i - a_i = \sum_{i=1}^{n} \lambda(J_i)$$

□

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Lemma 0.4

If I is a finite interval in $\mathcal{I}$ such that $I \subseteq \bigcup_{i=1}^{\infty} I_i$ where $I_i \in \mathcal{I}$, then $\lambda(I) \le \sum_{i=1}^{\infty} \lambda(I_i)$

proof

If I_i is infinite for some i, then, clearly,

$$\lambda(I) \le +\infty = \sum_{i=1}^{\infty} \lambda(I)$$

If I_i is finite for each i, then, we can assume that $I = [a,b]$ for $a,b \in \mathbb{R}$ and that each of ~~the~~ I_i is open. Then by Heine-Borel theorem, there exists some n such that $I \subseteq \bigcup_{i=1}^{n} I_i$.

Let $I_i = (a_i, b_i)$ for each $i = 1,2,\dots,n$. Since $a \in I$, there exists some i such that $a \in I_i$. We rename this interval as I_1.

If $b \in I_1$, then, by monotonicity of λ,

$$\lambda(I) \le \lambda(I_1) \le \sum_{i=1}^{n} \lambda(I_i) \le \sum_{i=1}^{\infty} \lambda(I_i)$$

and we are done. If $b \notin I_1$, then $b_1 < b$ and hence $b_i \in I_i$ for some $i \geq 2$. proceeding this way, we have $a_1 < a < b_1 < b_2 < \cdots < b_{m-1} < b < b_m$ for some $m \leq n$, and $a_i < b_{i-1}$ for each $i \geq 2$. Hence,

$$\begin{aligned}
\lambda(I) &= b-a \\
&< b_m - a_1 \\
&= \sum_{i=2}^{m} (b_i - b_{i-1}) + b_1 - a_1 \\
&\leq \sum_{i=2}^{m} (b_i - a_i) + b_1 - a_1 \\
&= \sum_{i=1}^{m} \lambda(I_i) \\
&\leq \sum_{i=1}^{\infty} \lambda(I_i)
\end{aligned}$$

In the general case when I is an arbitrary finite interval, given any $\varepsilon > 1$, we can find a closed interval $J \subset I$ such that $\varepsilon \lambda(J) = \lambda(I)$ and an open interval $J_i \supseteq I_i$ such that $\lambda(J_i) = \varepsilon \lambda(I_i)$ for every $i \geq 1$. Then, $J \subseteq \bigcup_{i=1}^{\infty} J_i$ and so, by the earlier case, $\lambda(J) \leq \sum_{i=1}^{\infty} \lambda(J_i)$. Therefore,

$$\begin{aligned}
\lambda(I) = \varepsilon \lambda(J) &\leq \varepsilon \sum_{i=1}^{\infty} \lambda(J_i) \\
&= \varepsilon^2 \sum_{i=1}^{\infty} \lambda(I_i)
\end{aligned}$$

Since this is true for all $\varepsilon > 1$, by letting $\varepsilon \to 1$ we have $\lambda(I) \leq \sum_{i=1}^{\infty} \lambda(I_i)$

□

Lemma 0.5

Let $I \in \underline{I}$ be a finite interval such that $I = \bigcup_{n=1}^{\infty} I_n$, where $I_n \in \underline{I}$ and $I_n \cap I_m = \phi$ for $n \neq m$. Then

$$\lambda(I) = \sum_{i=1}^{\infty} \lambda(I_n)$$

proof

First we note that, by lemma 0.4, $\lambda(I) \leq \sum_{n=1}^{\infty} \lambda(I_n)$.
To prove the reverse inequality, let $I = I(a,b)$ $a,b \in \mathbb{R}$. Let $I_i = I_i(a_i, b_i)$ for each $n \geq 1$.
Since $I_n \cap I_m = \phi$ $(n \neq m)$ we can assume (without loss of generality) that for all $k \geq 1$,
$a \leq a_1 < b_1 \leq a_2 < b_2 \leq \ldots < b_k \leq b$.
Then for every $k \geq 1$ we have

$$\sum_{n=1}^{k} \lambda(I_n) = \sum_{n=1}^{k} (b_n - a_n)$$

$$\leq b - a = \lambda(I)$$

Hence, $\sum_{n=1}^{\infty} \lambda(I_n) \leq \lambda(I)$.

Lemma 0.6

Let $I \in \underline{I}$ be any interval. Then

$$\lambda(I) = \sum_{n=-\infty}^{+\infty} \lambda(I \cap [n, n+1))$$

prof

If I is finite, then $I = \bigcup_{n=k}^{l} I \cap [n, n+1)$ for some integers k and l. And so, by lemma 0.5, we have the result.

If I is not finite, then $l(I) = \infty$ and for an infinite number of n's we will have $I \cap [n, n+1) = [n, n+1)$

Thus, $\sum_{n=-\infty}^{+\infty} l(I \cap [n, n+1)) = +\infty = l(I)$ □

Lemma 0.7 (countable additivity)

Let $I \in \mathcal{I}$ be any interval in $\mathbb{R}$ such that $I = \bigcup_{n=1}^{\infty} I_n$, $I_n \in \mathcal{I}$ and $I_n \cap I_m = \phi$ for $m \neq n$.

Then $l(I) = \sum_{n=1}^{\infty} l(I_n)$.

prof

If I is finite this is Lemma (5). If I is infinite, we use Lemma 0.5 and 0.6 to obtain the result. □

Lemma 0.8 (countable subadditivity)

Let $I \in \mathcal{I}$ be any interval such that $I \subseteq \bigcup_{n=1}^{\infty} I_n$, $I_n \in \mathcal{I}$. Then

$$l(I) \leq \sum_{n=1}^{\infty} l(I_n)$$

<u>proof</u>

If I is finite, then this is just Lemma 0.4.
If I is infinite and for all n, I_n is finite, then we write

$$I = \bigcup_{k=-\infty}^{+\infty} (I \cap [k, k+1))$$

and note that, for each k,

$$I \cap [k, k+1) \subseteq \bigcup_{n=1}^{\infty} (I_n \cap [k, k+1))$$

Using Lemmas 0.4 & 0.6, we have

$$\begin{aligned} l(I) &= \sum_{k=-\infty}^{+\infty} l(I \cap [k, k+1)) \\ &\leq \sum_{k=-\infty}^{+\infty} \sum_{n=1}^{\infty} l(I_n \cap [k, k+1)) \\ &= \sum_{n=1}^{\infty} \sum_{k=-\infty}^{\infty} l(I_n \cap [k, k+1)) \\ &= \sum_{n=1}^{\infty} l(I_n). \quad \square \end{aligned}$$

<u>Lemma 0.9</u> (Translation invariance)
For each $I \in \mathcal{I}$ and $x \in \mathbb{R}$, $l(I) = l(I+x)$ where $I + x = \{y + x \mid y \in I\}$.

1. GENERAL EXTENSION THEORY

1.1 First extension

Suppose $E \subseteq \mathbb{R}$ is a finite union of pairwise disjoint intervals, i.e

$E = \bigcup_{i=1}^{n} I_i$, $I_i \in \underset{\sim}{I}$ and $I_i \cap I_j = \phi$, $i \neq j$.

We define an extended length of E, denoted by $\tilde{\lambda}(E)$, by

$$\tilde{\lambda}(E) = \sum_{i=1}^{n} \lambda(I_i) = \sum_{i=1}^{n} (b_i - a_i).$$

$$\tilde{\lambda} : E \longrightarrow \sum_{i=1}^{n} \lambda(I_i) \in [0, \infty]$$

This function $\tilde{\lambda}$ is well-defined, for if, $E = \bigcup_{k=1}^{n} I_k = \bigcup_{l=1}^{m} J_l$, where $I_1, I_2, \dots, I_n$ are pairwise disjoint intervals and $J_1, J_2, \dots, J_m$ are pairwise intervals. Then $I_k = \bigcup_{k=1}^{n} (I_k \cap J_l)$ for each k, and $J_l = \bigcup_{k=1}^{n} (I_k \cap J_l)$ for each l, where the intervals $I_k \cap J_l$ are pairwise disjoint. Thus, using finite additivity of λ, we have

$$\sum_{k=1}^{n} \lambda(I_k) = \sum_{k=1}^{n} \sum_{l=1}^{m} \lambda(I_k \cap J_l) = \sum_{l=1}^{m} \lambda(J_l)$$

7/7/2023 Thus, we have a well-defined extended length function $\tilde{\lambda}$ a the class of subset a of $\mathbb{R}$ given by

$$F = \{E \subseteq \mathbb{R} \mid E \bigsqcup_{i=1}^{n} I_i,\ I_i \in \underset{\sim}{I}\}$$

Note that throughout the discussion we will use the notation $\overset{n}{\underset{i=1}{\sqcup}} I_i$ for the disjoint union of pointwise disjoint intervals $I_1, I_2, \ldots, I_n$.

Clearly, $I \in F$ for all $I \in \mathcal{I}$ and $\tilde{l}(I) = l(I)$. The function $\tilde{l} : F \longrightarrow [0, \infty]$ has the properties similar to the properties of l, that is, $\tilde{l}$ is non-negative, monotone, countably additive and countably subadditive.

In order to further extend the length function to a more general collection of subsets of $\mathbb{R}$, we first study the properties of $\mathcal{I}$ and F.

Theorem 1.1

The class $\mathcal{I}$ of all intervals in $\mathbb{R}$ satisfies the following properties:

i. $\phi, \mathbb{R} \in \mathcal{I}$

ii. If $I, J \in \mathcal{I}$, then $I \cap J \in \mathcal{I}$

iii. If $I \in \mathcal{I}$, then $I^c = J_1 \sqcup J_2$ with $J_1, J_2 \in \mathcal{I}$

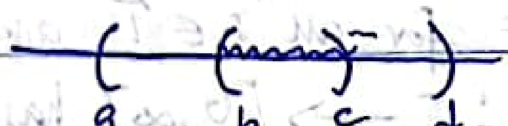
proof

i. $\phi = (a, a)$ $\forall a \in \mathbb{R}$, $\mathbb{R} = (-\infty, \infty)$

ii. Let $I, J \in \mathcal{I}$, then

Case 1

$I = (a, b)$, $J = (c, d)$, then either $I \cap J = \phi$, in which case we have $I \cap J \in \mathcal{I}$, or $I \cap J \neq \phi$, in which case we can assume without loss of generality, that $a \leq c < b \leq d$.

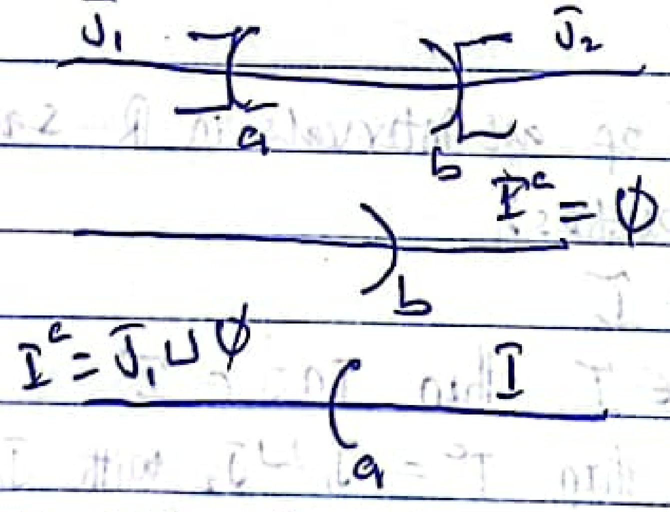


And so, $I \cap J$ will be an interval with left end point c and right end point b. Thus, $I \cap J \in \mathcal{I}$.

Case II

If I or J is infinite, then we can also observe as in case I that $I \cap J \in \mathcal{I}$.

iii) $I \in \mathcal{I}$, then



$I^c = \phi \sqcup J_2$

$I^c = J_1 \sqcup \phi$

$I = \mathbb{R}$, $I^c = \phi = \phi \sqcup \phi$ □.

(*) Theorem 1.2

The class $F = \{E \subseteq \mathbb{R} \mid E = \bigsqcup_{i=1}^{n} I_i,\ I_i \in \mathcal{I}\}$ has the following properties:

i. $\mathcal{I} \subseteq F$

ii. $\emptyset, \mathbb{R} \in F$.

iii. If $E, F \in F$, then $E \cap F \in F$.

iv. If $E \in F$, then $E^c \in F$.

v. F is the smallest class of subsets of $\mathbb{R}$ with the following properties (i), (ii), (iii) and (iv)

proof

(i) and (ii) are straight forward.

iii. Let $E, F \in F$. Then $E = \bigsqcup_{i=1}^{n} I_i$ and $F = \bigsqcup_{j=1}^{m} J_j$ where $I_i, J_j \in \mathcal{I}$. And so,

$$E \cap F = \left(\bigsqcup_{i=1}^{n} I_i\right) \cap \left(\bigsqcup_{j=1}^{m} J_j\right)$$
$$= \bigsqcup_{i=1}^{n} \bigsqcup_{j=1}^{m} (I_i \cap J_j)$$

which is a disjoint union of disjoint intervals $I_i \cap J_j \in \mathcal{I}$. Thus $E \cap F \in F$.

(iv). Let $E \in F$. Then, $E = \bigsqcup_{i=1}^{n} I_i$ where $I_i \in \mathcal{I}$. Now,

$$E^c = \left(\bigsqcup_{i=1}^{n} I_i\right)^c = \bigcap_{i=1}^{n} I_i^c.$$

By theorem 1.1 (ii), for each i,

$$I_i^c = J_1^i \sqcup J_2^i$$

where $J_1^i, J_2^i \in \mathcal{I}$.

Thus,

$$E^c = \bigcap_{i=1}^{n} \left(J_1^i \sqcup J_2^i\right)$$
$$= \bigsqcup_{k,l=1}^{2} \left(\bigsqcup_{i,j=1, i \neq j}^{n} \left(J_k^i \cap J_l^j\right)\right.$$

Which is a disjoint union of intervals and so, $E^c \in F$.

v. Let C_e be any collection of subsets of $\mathbb{R}$ such that $I \subseteq C_e$ and C_e has the properties (ii), (iii) and (iv). First, we note that if $E, F \in C_e$, then

$$E \cup F = (E^c \cap F^c)^c$$

So that $E \cup F \in C_e$

Now, let $E \in F$ then,

$E = \bigsqcup_{i=1}^{n} I_i$, $I_i \in I$. But as $I \subseteq C_e$, $I_i \in C_e$ for each i.

Thus $\bigsqcup_{i=1}^{n} I_i = E \in C_e$ showing that $F \subseteq C_e$. □

1.2 Semi-algebra and algebra of subset of a set

(*) **Definition (Semi-algebra of subsets):**

Let X be a non-empty set and let S be a collection of subsets of X. We say that S is a semi-algebra of subsets of X if it satisfies the following condition:

i. $\emptyset, X \in S$.

ii. $A, B \in S \Rightarrow A \cap B \in S$

iii. $A \in S \Rightarrow A^c = \bigsqcup_{i=1}^{n} C_i$, where $C_i \in S$

Examples

1. The collection $\mathcal{I}$ of all intervals in $\mathbb{R}$ is a semi-algebra of subsets of $\mathbb{R}$. This is proved in 1.1

2. Let $X = \mathbb{R}^2$ and $\mathcal{S} = \{I \times J \mid I, J \in \mathcal{I}\}$, the collection of all rectangles in $\mathbb{R}^2$. Then $\mathcal{S}$ is a semi-algebra of subsets of $\mathbb{R}^2$.

i. $\phi = \phi \times \phi$ and $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$

$\therefore \phi, \mathbb{R}^2 \in \mathcal{S}$

18/07/2023 ⊛ **Definition: (Algebra of subset of a set)**

Let X be a non-empty set and $\mathcal{A}$ be a collection of subsets of X. Then $\mathcal{A}$ is called an algebra of subsets of X if it satisfies the following: for all $A, B \in \mathcal{A}$

i. $\phi, X \in \mathcal{A}$

ii. $A, B \in \mathcal{A} \Rightarrow A \cap B \in \mathcal{A}$

iii. $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$

Example

① If X is any non-empty set, then the power set of X $P(X)$ is an algebra of subsets of X.

②. If $X = \mathbb{R}$, then the collection $F = \{E \subseteq \mathbb{R} \mid E = \bigcup_{j=1}^{n} I_j, I_i \in \mathcal{I}\}$ is an algebra of subsets of $\mathbb{R}$.

③ Let X be a any non-empty set. ~~The~~ the

the collection $A_e = \{E \subseteq X \mid$ either E is finite or E^c is finite$\}$ is an algebra of subsets of X.

proof

We show that A_e satisfies the three properties of the definition of algebra

i. since ϕ is finite it follows that $\phi, X \in A_e$

ii. Let $E, F \in A_e$. Then either E is finite or E^c is finite and either F is finite or F^c is finite. We consider two cases as follows:

case 1: If one of E or F is finite, then $E \cap F$ is finite and so is in A_e.

case 2: If both E and F are infinite, then both E^c and F^c are finite. thus, $(E \cap F)^c = E^c \cup F^c$ is finite and so, $E \cap F = A_e$

iii. Let $E \in A_e$, then either E or E^c is finite. If E is finite then $E^c \in A_e$. On the other hand If E is infinite then E^c is finite and so in A_e

Remark / Exercise

1. For any non-empty set X, a collection A_e of subsets of X is an algebra if and only if
i. $\phi, X \in A_e$
ii. $A \in A_e \Rightarrow A^c \in A_e$

$\Delta \to$ symmetric difference.
$\Lambda \to$ index set

iii. $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$. i.e
$A \cup B = (A^c \cap B^c)^c$ & $A \cap B = (A^c \cup B^c)^c$

② If $\mathcal{A}$ is an algebra of subsets of a set X then, for all $A, B \in \mathcal{A}$,
$A \Delta B = A \setminus B \cup B \setminus A \in \mathcal{A}$.

③ If $\mathcal{A}$ is an algebra of subsets of a set X and $\bigcup_{i=1}^{\infty} E_i \in \mathcal{A}$, where $E_1, E_2, \dots \in \mathcal{A}$, then there exists $F_1, F_2, \dots \in \mathcal{A}$ such that $F_i \cap F_j = \emptyset$ for all $i \neq j$ and $\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} F_i$

proof

Define, for each $n \geq 2$, $F_n = E_n \setminus \bigcup_{i=1}^{n-1} E_i$ and $F_1 = E_1$

~~Given~~

Given any family $\{\mathcal{A}_j\}_{j \in \Lambda}$ of algebra of a set X, the intersection $\mathcal{A} := \bigcap_{j \in \Lambda} \mathcal{A}_j$ is also an algebra of subsets of X. To see this we note that:

i. $\emptyset, X \in \mathcal{A}$ since $\emptyset, X \in \mathcal{A}_j; \forall j \in \Lambda$

ii. $A, B \in \mathcal{A} \Rightarrow A, B \in \mathcal{A}_j; \forall j \in \Lambda$
$\Rightarrow A \cap B \in \mathcal{A}_j$; for all $j \in \Lambda$
$\Rightarrow A \cap B \in \mathcal{A}$

iii. $A \in \mathcal{A} \Rightarrow A \in \mathcal{A}_j; \forall j \in \Lambda$
$\Rightarrow A^c \in \mathcal{A}_j; \forall j \in \Lambda$
$\Rightarrow A^c \in \mathcal{A}$

Now, given any collection $\mathcal{C}$, namely the power

$P(X)$. And so, the class of all algebras of subsets of X containing $\mathcal{C}$, given by,

$\{\mathcal{A} \mid \mathcal{A}$ is an algebra of X and $\mathcal{C} \subseteq \mathcal{A}\}$ is non empty.

Hence

$\cap\{\mathcal{A} \mid \mathcal{A}$ is an algebra of X with $\mathcal{C} \subseteq \mathcal{A}\}$ is also an algebra of subsets of X containing $\mathcal{C}$. It is the smallest such algebra of subsets of X containing $\mathcal{C}$. Thus we have proved,

Theorem 1.4

Let $\mathcal{C}$ be any collection of subsets of a set X. Then there is a unique algebra of subsets of X $\mathcal{B}$ such that $\mathcal{C} \subseteq \mathcal{B}$ and whenever $\mathcal{A}$ is an algebra of subset of X with $\mathcal{C} \subseteq \mathcal{A}$, $\mathcal{B} \subseteq \mathcal{A}$.

Definition

Let X be a non-empty set and $\mathcal{C}$ be any collection of subsets of X. Then the algebra generated by $\mathcal{C}$, denoted by $\mathcal{A}(\mathcal{C})$, is the smallest algebra of subsets of X containing $\mathcal{C}$, that is, $\mathcal{A}(\mathcal{C}) = \cap\{\mathcal{A}$ is an algebra of X and $\mathcal{C} \subseteq$

Examples

1. Let $X = \mathbb{R}$ and $\mathcal{C}_e = \mathcal{I}$. Then
$A_e(\mathcal{I}) = \{E \subseteq \mathbb{R} / E = \bigcup_{i=1}^{n} I_i ; I_i \in \mathcal{I}\} = F(\mathcal{I})$.
This fact was proven in theorem 1.2

2. Let X be any non-empty set and $\mathcal{C}_e = \{\{x\} / x \in X\}$
Then, $A_e(\mathcal{C}_e) = \{E \subseteq X / E$ is finite or E^c is finite$\}$.

Proof

Recall that, we already proved $A_e(\mathcal{C}_e)$ to be an algebra of subsets of X. It remains to show that:
i. $\mathcal{C}_e \subseteq A_e(\mathcal{C}_e)$
ii. $A_e(\mathcal{C}_e)$ is the smallest algebra with the property (i)
So, (i) Since, for each $x \in X$, $\{x\}$ is finite, $\{x\} \in A_e(\mathcal{C}_e)$ and so, $\mathcal{C}_e \subseteq A_e(\mathcal{C}_e)$.

ii. Let $\mathcal{B}$ be any other algebra of subsets of X. with $\mathcal{C}_e \subseteq \mathcal{B}$. Let $E \in A_e(\mathcal{C}_e)$. Then either E is finite or E^c is finite. If E is finite, then
$E = \bigcup_{i=1}^{n} \{x_i\}$ where $x_i \in X$.
But $\{x_i\} \in \mathcal{C}_e \subseteq \mathcal{B}$ and so, as $\mathcal{B}$ is an algebra and E a finite union of subsets in $\mathcal{B}$, it follows that $E \in \mathcal{B}$.
Thus, $A_e(\mathcal{C}_e) \subseteq \mathcal{B}$
If E^c is finite, then $E^c = \bigcup_{i=1}^{m} \{y_i\}$ where $y_i \in X$
But by similar observations as shown above, we have $E^c \in \mathcal{B}$ and so $E \in \mathcal{B}$. Thus, again $A_e(\mathcal{C}_e) \subseteq \mathcal{B}$.

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Remark

Note that while it is easy in the last two examples to describe the algebra generated by the given collections of subsets, it is not generally possible to give an easy description of algebra generated for any collection $\mathcal{C}$ of subsets of a given set. But if the collection $\mathcal{C}$ is as semi-algebra, then such a description of the algebra generated is possible.

(*) Theorem 1.5

Let $\mathcal{S}$ be a semi algebra of a non-empty set X. Then the algebra $\mathcal{A}(\mathcal{S})$, generated by $\mathcal{S}$, is given by

$$\mathcal{A}(\mathcal{S}) = \{E \subseteq X \mid E = \bigsqcup_{i=1}^{n} C_i, \; C_i \in \mathcal{S}\}$$

proof

Let $\mathcal{A} = \{E = \{E \in X \mid E = \bigsqcup_{i=1}^{n} C_i, C_i \in \mathcal{S}\}$. Then

i. Since every $C \in \mathcal{S}$ is a disjoint union of itself. $\mathcal{S} \subseteq \mathcal{A}$. Thus, $\phi, X \in \mathcal{A}$

ii. Let $E \in \mathcal{A}$. Then $E = \bigsqcup_{i=1}^{n} C_i$, $C_i \in \mathcal{S}$. Since $\mathcal{S}$ is a semi-algebra, $C_i^c = \bigsqcup_{j=1}^{m} A_j^i$, $A_j^i \in \mathcal{S}$. Therefore, we have, $E^c = \bigcap_{i=1}^{n} \left(\bigsqcup_{j=1}^{m} A_j^i\right) = \bigsqcup (A_j^i \cap A_l^k)$, a finite ~~iii Let $E, F \in \mathcal{A}$. Then algebra~~ disjoint union of members in $\mathcal{S}$. Thus, $E^c \in \mathcal{A}$.

iii. Let $E, F \in A_e$. Then, $E = \biguplus_{i=1}^{n} C_i$ and $F = \biguplus_{j=1}^{m} D_j$ with $C_i; D_j \in S$. But then,

$$E \cap F = \left(\biguplus_{i=1}^{n} C_i\right) \cap \left(\biguplus_{j=1}^{m} D_j\right) = \biguplus_{i,j}^{n} (C_i \cap D_j)$$

Which is a finite disjoint union of subset in S and so $E \cap F \in A_e$.

Now, let B be any algebra of X such that $S \subseteq B$. If $E \in A_e$, then $E = \biguplus_{i=1}^{n} C_i$, $C_i \in S \subseteq B$. Since B is an algebra, it is closed under finite union, Thus E being a finite union of member of B must be in B. Therefore, $A_e \subseteq B$

Hence, $A_e = A_e(S)$, the algebra generated by S. ∎

Theorem 1.6

Let $\mathcal{C}$ be any collection of subsets of a set X and let $E \subseteq X$. If $\mathcal{C} \cap E := \{C \cap E \mid C \in \mathcal{C}\}$. Then $A_e(\mathcal{C}) \cap E = A_e(\mathcal{C} \cap E)$.

<u>Proof</u> (Exercise)

~~First, we note~~

Definition

Let X be a non-empty set and let Σ be a class of subsets of X. Then Σ is called σ-algebra if it has the following properties:

i. $\emptyset, X \in \Sigma$.

ii. If $A \in \Sigma$ then $A^c \in \Sigma$

iii. If $A_1, A_2, \ldots, \in \Sigma$, then $\bigcup_{i=1}^{\infty} A_i \in \Sigma$

(8) **Definition**

Let X be a non-empty and Σ be a σ-algebra. Then, the pair (X, Σ) is called measurable space.

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Examples

1. Every σ-algebra is an algebra

2. It is not true that every algebra is a σ algebra. To see this we consider a non-empty set uncountable set X and $A_e = \{E \subseteq X \mid E$ is finite or E^c is finite$\}$. This collection A_e was proved to be an algebra. We now see that it is not a σ-algebra. Since X is uncountable, it follows that there exists a non-empty ~~set~~ countably infinite subset E of X with E^c infinite for ~~that of~~ otherwise, $X = E \cup E^c$ will be

be countable to the assumption, thus

$E = \{x_1, x_2, x_3, \ldots\} = \bigcup_{i=1}^{\infty} \{x_i\}$

But each of $\{x_i\} \in A_e$ with $E \notin A_e$. Thus, A_e

With $E \notin A$. Thus is not a σ-algebra.

B. For any set X the collections $\{\emptyset, X\}$ and $P(X)$ are σ-algebra chaubtic.

H. Let X be an uncountable set and $\Sigma = \{E \subseteq X \mid E$ is countable or E^c is countable$\}$. Then Σ is a σ-algebra.

proof

Clearly (i) $\emptyset \in \Sigma$.

and (ii) if $E \in \Sigma$ then $E^c \in \Sigma$

iii. Let $E_1, E_2, \ldots \in \Sigma$. Then either

Case I: E_i is countable for all i, in which case, we have $\bigcup_{i=1}^{\infty} E_i$ to be countable also. Thus, $\bigcup_{i=1}^{\infty} E_i \in \Sigma$ or

Case II: E_j is uncountable for some j, in which case, E_j^c is countable. But then $\bigcap_{i=1}^{\infty} E_i^c \subseteq E_j^c$ is again countable. Thus,

$$\left(\bigcup_{i=1}^{\infty} E_i\right)^c = \bigcap_{i=1}^{\infty} E_i^c$$

is countable and so $\bigcup_{i=1}^{\infty} E_i \in \Sigma$

Hence, Σ is a σ-algebra. □

Theorem 1.7

If X is any non-empty set and $\mathcal{C}$ is any collection of subsets of X, then the collection $\bigcap_{\mathcal{C} \subseteq \Sigma} \{\Sigma \mid \Sigma \text{ is a } \sigma\text{-algebra subsets of } X\}$ is a again σ-algebra and containing $\mathcal{C}$. It is the smallest σ-algebra containing $\mathcal{C}$.

(2) Definition

Let X be a non-empty set and $\mathcal{C}$ a collection of subsets of X. The σ-algebra generated by $\mathcal{C}$ denoted by $\sigma(\mathcal{C})$, is the intersection of all σ-algebra of subsets of X containing $\mathcal{C}$.

Example

Let X be a non-empty uncountable set and but $\mathcal{C} = \{\{x\} \mid x \in X\}$ then $\sigma(\mathcal{C}) = \{E \subseteq X \mid E$ is countable or E^c is countable$\}$.

Proof

Let $\Sigma = \{E \subseteq X \mid E$ is countable or E^c is countable$\}$. We have already seen that Σ is a σ-algebra. Also $\{x\} \in \Sigma$ for all $x \in X$ so that $\mathcal{C} \subseteq \Sigma$. It remains to show that Σ is the smallest σ-algebra containing $\mathcal{C}$.

For this, we let $\mathcal{B}$ be any σ-algebra of subsets of X such that $\mathcal{C} \subseteq \mathcal{B}$.

Let $E \in \mathcal{E}$. Then, either E is countable or E^c is countable.
In the former, we have $E = \bigcup_{i=1}^{\infty} \{x_i\}$. But since each $\{x_i\} \in \mathcal{C} \subseteq \mathcal{B}$ and $\mathcal{B}$ is a δ- algebra we have $E \in \mathcal{B}$.
In the latter $E^c = \bigcup_{i=1}^{\infty} \{y_i\}$ with each $\{y_i\} \in \mathcal{C} \subseteq \mathcal{B}$ and ~~there~~ $\mathcal{B}$. a δ- algebra. Thus, $E^c \in \mathcal{B}$ and so $E \in \mathcal{B}$. Hence, $\mathcal{E} \subseteq \mathcal{B}$ and therefore, $\mathcal{E} = \delta(\mathcal{C})$

Remark

It is not in general possible to always describe the δ- algebra generated by a given collection $\mathcal{C}$ of subsets of a set X. However, special interest is given to the case when X is a topological space and when $\mathcal{C}$ is either the collection of open sets of X or the collection of closed - subsets of X.

Definition

Let X be a topological space and τ be its topology. Then, the δ- algebra $\delta(\tau)$ (the δ- algebra generated by τ) is called the Borel δ-algebra of subsets of X.
Members of $\delta(\tau)$ are called Borel subsets of X.

(8) **Lemma 1.8**

Let X be a topological space with topology τ and let $\mathcal{C}$ be the collection of all closed subsets of X. Then

$$\delta(\tau) = \delta(\mathcal{C})$$

proof

Let $A \in \tau$

$\Rightarrow A$ is open in X

$\Rightarrow A^c$ is closed in X

$\Rightarrow A^c \in \mathcal{C} \subseteq \delta(\mathcal{C})$.

And so, $\tau \subseteq \delta(\mathcal{C})$. Hence $\delta(\tau) \subseteq \delta(\mathcal{C})$

Similarly, $\delta(\mathcal{C}) \subseteq \delta(\tau)$. ▫

Lemma 1.9

Let X be a non-empty set and $\mathcal{C}$ be a collection of subsets of X. Then

$$\delta(A_c(\mathcal{C})) = \delta(\mathcal{C})$$

proof

We note that $\mathcal{C} \subseteq A_c(\mathcal{C}) \subseteq \delta(A_c(\mathcal{C}))$

$\Rightarrow \delta(\mathcal{C}) \subseteq \delta(A_c(\mathcal{C}))$.

Also, $\mathcal{C} \subseteq \delta(\mathcal{C}) \Rightarrow A_c(\mathcal{C}) \subseteq \delta(\mathcal{C})$

$\Rightarrow \delta(A_c(\mathcal{C})) \subseteq \delta(\mathcal{C})$. Thus,

$\delta(A_c(\mathcal{C})) = \delta(\mathcal{C})$.

Lemma 1.10

If $Y \subseteq X$, then $\delta(\mathcal{C} \cap Y) = \delta(\mathcal{C}) \cap Y$.

Definition

If Σ is a δ-algebra of a set X, then member of Σ is called measurable subset.

31/07/2023 **Definition**

Let X be a non-empty set and M be a class of subsets of X. We say that M is a monotone class if

show that

i. $\bigcup_{n=1}^{\infty} A_n \in M$ whenever $A_1, A_2, \ldots \in M$ and $A_n \subseteq A_{n+1}$ for all $n \geq 1$.

ii. $\bigcap_{n=1}^{\infty} A_n \in M$ whenever $A_1, A_2, \ldots \in M$ and $A \supseteq A_{n+1}$ for all $n \geq 1$

~~Examples ① Every δ-algebra of subset of a set X. Then~~

Examples ⓪ $P(X)$ is a monotone class

① Every δ-algebra is a monotone class

Proof

Let M be a δ-algebra of subset of a set X. Then

If $A_1, A_2, \ldots \in M$ such that $A_n \subseteq A_{n+1}$ for all $n \geq 1$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$ since $\mathcal{M}$ is a σ-algebra.

ii. If $A_1, A_2, \ldots \in \mathcal{M}$ such that $A_n \supseteq A_{n+1}$ for all $n \geq 1$, then $A_n^c \in \mathcal{M}$ for all $n \geq 1$ as $\mathcal{M}$ is a σ-algebra.

But $A_n^c \subseteq A_{n+1}^c$ for all $n \geq 1$ so that by part (i)

$$\bigcup_{n=1}^{\infty} A_n^c \in \mathcal{M} = \left(\bigcap_{n=1}^{\infty} A_n\right)^c$$

And so, as $\mathcal{M}$ is a σ-algebra $\bigcap_{n=1}^{\infty} A_n \in \mathcal{M}$

Thus, $\mathcal{M}$ is a monotone class. □

② Let X be an uncountable set and $\mathcal{M} = \{A \subseteq X \mid A \text{ is countable}\}$. Then $\mathcal{M}$ is a monotone class which is not a σ-algebra.

proof

Let $A_1, A_2, \ldots \in \mathcal{M}$

~~i. If $A_n \subseteq A_{n+1}$ f~~

i. Suppose $A_n \subseteq A_{n+1}$ for all $n \geq 1$. Then since $A_n \in \mathcal{M}$ for all $n \geq 1$, A_n is countable for all $n \geq 1$. But then, $\bigcup_{n=1}^{\infty} A_n$ is countable so that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$.

ii. Suppose $A_n \supseteq A_{n+1}$. Then $\bigcap_{n=1}^{\infty} A_n \subseteq A_1$ which is countable being in $\mathcal{M}$. Thus, $\bigcap_{n=1}^{\infty} A_n$ is countable and so in $\mathcal{M}_e$.

Note that, for all $A \in \mathcal{M}_e$, $A^c \notin \mathcal{M}_e$, for otherwise, we will have both A and A^c to be countable which contradict the fact that X is uncountable

$$X = A \cup A^c \quad \square$$

· Given any collection $\mathcal{C}_e$ of subsets of X, there is always a monotone class containing $\mathcal{C}_e$, namely $P(x)$. Thus, we have:

Definition

If $\mathcal{C}_e$ is any collection of subset of X, then the smallest monotone class of X containing $\mathcal{C}_e$ is called the monotone class of X generated by $\mathcal{C}_e$ and it is denoted by $\mathcal{M}_e(\mathcal{C}_e)$.

Theorem 1.11

Let X be a non-empty set. Then

(i) Intersection of any two monotone classes of X is a monotone class.

(ii) For any collection $\mathcal{C}_e$ of subsets of X,

$$\mathcal{M}_e(\mathcal{C}_e) = \bigcap \mathcal{M}_e. \; \mathcal{M}_e \text{ monotone class, } \mathcal{C}_e \subseteq \mathcal{M}_e.$$

Lemma 1.12

Let $\mathcal{C}$ be any collection of subsets of X. Then

i. If $\mathcal{C}$ is an algebra and a monotone class, then $\mathcal{C}$ is a σ-algebra

ii. $\mathcal{C} \subseteq \mathcal{M}_c(\mathcal{C}) \subseteq \sigma(\mathcal{C})$

proof

i. It remains to show that $\mathcal{C}$ is closed under countable union. For this let $A_1, A_2, \ldots \in \mathcal{C}$. Then $\bigcup_{i=1}^{n} A_i \in \mathcal{C}$ since $\mathcal{C}$ is an algebra. But, now A_1, $A_1 \cup A_2, \ldots \bigcup_{i=1}^{n} A_i, \ldots \in \mathcal{C}$ is a monotone increasing sequence. Thus $\bigcup_{n=1}^{\infty}\left(\bigcup_{i=1}^{n} A_i\right) \in \mathcal{C}$ i.e. $\bigcup_{i=1}^{\infty} A_n \in \mathcal{C}$. Hence $\mathcal{C}$ is a σ-algebra.

ii. The first inclusion is immediate from definition. For the second inclusion $\sigma(\mathcal{C})$ is a monotone class (by example 1). Thus, $\mathcal{M}_c(\mathcal{C}) \subseteq \sigma(\mathcal{C})$. □

Theorem 1.13 (σ-algebra monotone class theorem)

Let X be a non-empty set and $\mathcal{A}$ be an algebra of subsets of X. Then $\sigma(\mathcal{A}) = \mathcal{M}_c(\mathcal{A})$.

proof

By lemma 1.12 (ii), $\mathcal{M}_c(\mathcal{A}) \subseteq \sigma(\mathcal{A})$. To prove the reverse inclusion, it is enough, by lemma 1.12 (i), to show that $M_e(A_e)$ is an algebra.

i. Since $A_e \subseteq M_e(A_e)$, it follow that $\emptyset, X \in M_e(A_e)$ as A_e is an algebra.

ii. Let $B := \{E \subseteq X \mid E^c \in M_e(A_e)\}$. To prove that $M_e(A_e)$ is closed under compliment it enough to show that $M_e(A_e) \subseteq B$. We prove this by showing B is a monotone class containing A_e.

As A_e is an algebra and $A_e \subseteq M_e(A_e)$, we have $A_e \subseteq B$.

Let $E_1, E_2, \dots \in B$ be such that $E_n \uparrow$ for all $n \geq 1$. Then $E_n^c \in M_e(A_e)$ for all $n \geq 1$. And so, $\bigcap_{n=1}^{\infty} E_n^c \in M_e(A_e)$ as $E_n^c \downarrow$ and $M_e(A_e)$ is a monotone class.

Therefore, $\left(\bigcup_{n=1}^{\infty} E_n\right)^c = \bigcap_{n=1}^{\infty} E_n^c \in M_e(A_e)$ implies $\bigcup_{n=1}^{\infty} E_n \in B$

Next, let $F_1, F_2, \dots \in B$ be such that $F_n \downarrow$ for all $n \geq 1$. Then, $F_n^c \in M_e(A_e)$ for all $n \geq 1$. And so, $\bigcup_{n=1}^{\infty} F_n^c \in M_e(A_e)$ as $F_n^c \uparrow$ as $M_e(A_e)$ is a monotone class. Thus,

$$\left(\bigcap_{n=1}^{\infty} F_n\right)^c = \bigcup_{n=1}^{\infty} F_n^c \in M_e(A_e) \text{ so that}$$

$$\bigcap_{n=1}^{\infty} F_n \in B$$

Therefore, B is a monotone class containing A_e.

Hence, $M_e(A_e) \subseteq B$ ✓

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ii Let $F \in M_e(A_e)$ be a fixed subset and $\mathcal{L}(F) := \{E \subseteq X \mid E \cup F \in M_e(A_e)\}$.

To show that $M_e(A_e)$ is closed under finite union it is enough to show that $M_e(A_e) \subseteq \mathcal{L}(F)$.

We prove this by showing that $\mathcal{L}(F)$ is a monotone class containing A_e. Then we will have that for all $E, F \in M_e(A_e) \Rightarrow \mathcal{L}(F) \Rightarrow E \in \mathcal{L}(F)$

$\Rightarrow E \cup F \in M_e(A_e)$

so that $M_e(A_e)$ is closed under finite union.

We approach proving the inclusion $M_e(A_e) \subseteq \mathcal{L}(F)$ by showing that $\mathcal{L}(F)$ is a monotone class containing A_e.

Let $E_n \uparrow$ with $E_n \in \mathcal{L}(F)$ $\forall n$. Then $E_n \cup F \in M_e(A_e)$ for all n.

$\Rightarrow \bigcup_{n=1}^{\infty} (E_n \cup F) = (\bigcup_{n=1}^{\infty} E_n) \cup F \in M_e(A_e)$

$\Rightarrow \bigcup_{n=1}^{\infty} E_n \in \mathcal{L}(F)$.

Similarly, if $E_n \downarrow$ with $E_n \in \mathcal{L}(F)$ for all n, we have that $\bigcap_{n=1}^{\infty} E_n \in \mathcal{L}(F)$.

Thus, $\mathcal{L}(F)$ is a monotone class.

Now, note that for all $E, F \in M_e(A_e)$,

$E \in \mathcal{L}(F) \Leftrightarrow F \in \mathcal{L}(E)$. —— ①

~~$E \cup F \in M_e(A_e)$~~ —— ①

If $F \in A_e$, then $\forall E \in A_e$ we have $E \cup F \in A_e$

$\Rightarrow E \in \mathcal{L}(F). \Rightarrow \mathcal{A}_e \subseteq \mathcal{L}(F) \ \forall F \in \mathcal{A}_e$

$\Rightarrow \mathcal{M}_e(\mathcal{A}_e) \subseteq \mathcal{L}(F) \ \forall F \in \mathcal{A}_e.$

Thus, by eqn-①, we have

$E \in \mathcal{L}(F) \ \forall E \in \mathcal{M}_e, \ \forall F \in \mathcal{A}_e$

$F \in \mathcal{L}(E) \ \forall E \in \mathcal{M}_e(\mathcal{A}_e), \ \forall F \in \mathcal{A}_e$

$\Rightarrow E \in \mathcal{L}(F) \ \forall E \in \mathcal{M}_e(\mathcal{A}_e)$

$\Rightarrow \mathcal{M}_e(\mathcal{A}_e) \subseteq \mathcal{L}(F).$

1.3 Set Functions

Definition:

Let $\mathcal{C}$ be a collection of subsets of a set X. A function $\mu : \mathcal{C} \to [0, \infty]$ is called a set function.

$$\mu : A \longmapsto \mu(A) \in [0, \infty]$$

Definition

A set function $\mu : \mathcal{C} \to [0, \infty]$ is said to be:

i. monotone: If for all $A, B \in \mathcal{C}$, $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$.

ii. Finitely additive: If for all $A_1, A_2, \ldots, A_n \in \mathcal{C}$ such that $A_i \cap A_j = \emptyset$, $i \neq j$,

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i).$$

iii. Countably additive: If for all $A_1, A_2, \dots \in \mathcal{C}$ such that $A_i \cap A_j = \phi$ $i \neq j$

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

iv. Countably subadditive: If whenever $A \subseteq \bigcup_{i=1}^{\infty} A_i$ with $A, A_i \in \mathcal{C}$ for all i, then $\mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i)$.

V. Measure: If $\phi \in \mathcal{C}$, $\mu(\phi) = 0$ and μ is countably additive.

Examples

1. The length function $\lambda: \mathcal{I} \to [0, \infty]$ and its $\lambda: F = \mathcal{A}(\mathcal{I}) \to [0, \infty]$ are all measures.

2. Let $X = \{x_n \mid n = 1, 2, \dots\}$ and $\{p_n\}_{n \geq 1}$ be a sequence of non-negative real numbers. For any subset $A \subseteq X$, define $\mu: \mathcal{P}(X) \to [0, \infty]$ by

$$\mu(A) := \begin{cases} 0, & \text{if } A = \phi \\ \sum_{\{i \mid x_i \in A\}} p_i & \text{if } A \neq \phi. \end{cases}$$

Then μ is a measure, called the discrete measure with 'mass' p_i at x_i.

If $\mu(X) = \sum_i p_i = 1$, μ is called discrete probability distribution. $\mu(\{x_i\}) = p_i$

Proof

Let $A = \bigcup_{i=1}^{\infty} A_i$. Without loss of generality we may assume that each $A_i = \{x_{k_i}\}$. Then, by definition

$$\mu(A) = \sum_{i=1}^{\infty} P_{k_i} = \lim_{n\to\infty}\sum_{i=1}^{n} P_{k_i} = \lim_{n\to\infty}\left(\sum_{i=1}^{n}\mu(A_i)\right) = \sum_{i=1}^{\infty}\mu(A_i) \quad \square$$

Remark.

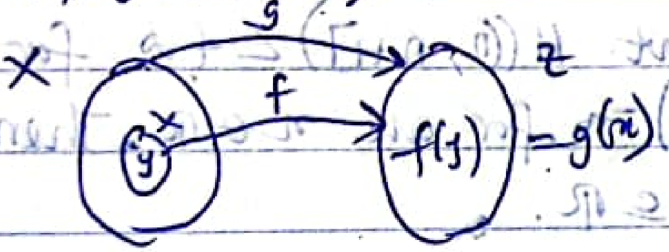
i. If $X = \{0,1,2,\dots\}$ and $0<\alpha<1$ with $P_k := \binom{n}{k}\alpha^k(1-\alpha)^{n-k}$, $0 \le k \le n$. Then μ is called Binomial distribution.

ii. If $X = \{0,1,2,\dots\}$ and $P_k := \frac{\lambda^k e^{-\lambda}}{k!}$, $k = 0,1,2,\dots$ and $\lambda > 0$. μ is called poisson distribution.

7/08/2023

Definition

A function $g : X \to Z$ is said to be an extension of a function $f : Y \subseteq X \to Z$ if, for $y \in Y$, $g(y) = f(y)$.



Example

The function $\bar{\lambda} : A_0(\mathcal{I}) \to [0,\infty]$
$\bar{\lambda}\left(\bigcup_{i=1}^{n} I_i\right) = \sum_{i=1}^{n} \bar{\lambda}(I_i)$ $\left\{\bigcup_{i=1}^{n} I_i \mid I_i \in \mathcal{I}\right\}$ is an extension of the length fn $\lambda : \mathcal{I} \to [0,\infty]$.

In the next theorem we give a general version of the extension of the length function.

Theorem 1.14
Given a measure μ defined on a semialgebra $\mathcal{S}$, there is a unique extension $\bar{\mu}$ of μ on $\mathcal{A}(\mathcal{S})$ such that $\bar{\mu}$ is a measure. $\{\bigcup_{i=1}^{n} E_i \mid E_i \in \mathcal{S}\}$

proof

$$\bar{\mu}(A) = \sum_{i=1}^{n} \mu(E_i)$$

The next theorem implies the impossibility of extending the length function to all subsets of $\mathbb{R}$.

Theorem 1.15 (S.M. Ulam, 1930)
Let μ be a measure defined on all subsets of $\mathbb{R}$ such that $\mu((n, n+1]) < +\infty$ for all $n \in \mathbb{Z}$ and $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}$. Then $\mu(E) = 0$ for all $E \subseteq \mathbb{R}$. ▫

⊛ Theorem 1.16 (properties of set fn on algebra).
Let $\mathcal{A}$ be an algebra of subsets of X and let $\mu : \mathcal{A} \to [0, \infty]$ be a set function. Then:
i. If μ is finitely additive and, for some $B \in \mathcal{A}$, $\mu(B) < +\infty$, then ~~$\mu(B) = \mu(B)$~~
$\mu(B \setminus A) = \mu(B) - \mu(A)$ for all $A \subseteq B$ in $\mathcal{A}$
ii. If μ is finitely additive, then μ is monoton

iii. Suppose $\mu(\emptyset)=0$. Then μ is countably additive if and the only if μ is both finitely-additive and countably subadditive.

Proof

(i) Let $A, B \in \mathcal{A}$ with $A \subseteq B$ and $\mu(B) < +\infty$. Then

$$B = A \cup B \setminus A$$

$$\Rightarrow \mu(B) = \mu(A \cup B \setminus A)$$

$$= \mu(A) + \mu(B \setminus A)$$

(ii) Suppose μ is finitely additive and let $A \subseteq B$ for $A, B \in \mathcal{A}$. Then

$$B = A \cup B \setminus A$$

$$\rightarrow \mu(B) = \mu(A \cup B \setminus A)$$

$$= \mu(A) + \mu(B \setminus A)$$

$$\geq \mu(A)$$

Thus, μ is monotone.

(iii) Suppose $\mu(\emptyset)=0$ and let μ be countably additive. Then let

$A = \biguplus_{i=1}^{n} A_i$, $A_i \in \mathcal{A}$. Then

$$A = \biguplus_{i=1}^{n} A_i = \biguplus_{i=1}^{\infty} A_i \text{ where}$$

$A_i = \emptyset$ for $i > n$. By countable additivity of

$$\mu(A) = \sum_{i=1}^{\infty} \mu(A_i) = \sum_{i=1}^{n} \mu(A_i) + \sum_{i=n+1}^{\infty} \mu(\emptyset) = \sum_{i=1}^{n} \mu(A_i)$$

and so μ is finitely additive

Next, let $A \subseteq \bigcup_{i=1}^{\infty} A_i$, $A_i \in \mathcal{A}_0$

then,

$$\bigcup_{i=1}^{\infty} A_i = \bigsqcup_{i=1}^{\infty} B_i$$

where $B_1 = A_1$, $B_n = A_n \setminus \bigcup_{i=1}^{n-1} A_i$

Thus, by countable additivity of μ,

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \mu\left(\bigsqcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} \mu(B_i) \le \sum_{i=1}^{\infty} \mu(A_i).$$

Since we have proved μ to be finitely additive and by part (ii) μ is monotone.

Now, again as μ is monotone,

$$\mu(A) \le \mu\left(\bigcup_{i=1}^{\infty} A_i\right) \le \sum_{i=1}^{\infty} \mu(A_i).$$

Thus, μ is countably subadditive

Conversely, suppose $\mu(\emptyset) = 0$ and μ is both finitely additive and countably sub-additive.

Let $A \in \mathcal{A}_0$ with $A = \bigsqcup_{i=1}^{\infty} A_i$, $A_i \in \mathcal{A}_0$. Then since μ is countably subadditive,

$$\mu(A) \le \sum_{i=1}^{\infty} \mu(A_i)$$

To show that μ is countably additive we only need to show that $\mu(A) \ge \sum_{i=1}^{\infty} \mu(A_i)$.

Now,

$A = \bigsqcup_{i=1}^{\infty} A_i \Rightarrow \bigsqcup_{i=1}^{n} A_i \subseteq A$ for all n.

Since μ is finitely additive, by (ii) μ is monotone and so

$$\mu\left(\bigsqcup_{i=1}^{n} A_i\right) \le \mu(A) \quad \forall n$$

$$\sum_{i=1}^{n} \mu(A_i) \le \mu(A) \quad \forall n$$

$$\Rightarrow \sum_{i=1}^{\infty} \mu(A_i) \le \mu(A) \quad \square$$

(*) <u>Definition</u>

A set function $\mu : \mathcal{A} \to [0, \infty]$ is said to be continuous from below if for all increasing sequence $A_n \uparrow$ in $\mathcal{A}$,

$$\mu\left(\bigcup_{i=1}^{\infty} A_n\right) = \lim_{n \to \infty} \mu(A_n).$$

μ is said to be continuous from above if, for all decreasing sequence $A_n \downarrow$ in $\mathcal{A}$,

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} \mu(A_n).$$

(*) Theorem 1.17

Let $\mathcal{A}$ be an algebra of subsets of a set X and $\mu : \mathcal{A} \to [0, \infty]$ be a finitely additive set function such that $\mu(\emptyset) = 0$. Then μ is a measure if and only if (μ is) continuous from above

proof

Suppose μ is a measure. Then μ is countably additive.

Let $A = \bigcup_{n\geq 1}^{\infty} A_n$ where $A \in \mathcal{A}$,

$A_n \in \mathcal{A}$ with $A_n \subseteq A_{n+1}$ for all n.

Define $B_1 = A_1$, $B_n = A_n \setminus \bigcup_{i=1}^{n-1} A_i$ for $n \geq 2$.

Then $B_n \in \mathcal{A}$; $B_n \cap B_m = \phi$ for all $n \neq m$ and $\bigcup_{n=1}^{\infty} A_n = \bigsqcup_{n=1}^{\infty} B_n$.

And so,

$$\mu(A) = \mu\left(\bigsqcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n) = \lim_{k\to\infty} \sum_{n=1}^{k} \mu(B_n)$$

$$= \lim_{k\to\infty} \mu\left(\bigsqcup_{n=1}^{k} B_n\right)$$

$$= \lim_{k\to\infty} \mu\left(\bigcup_{n=1}^{k} A_n\right)$$

$$= \lim_{k\to\infty} \mu(A_k) \text{ as } A_n \uparrow$$

thus, μ is continuous from below.

Conversely, suppose μ is continuous from below. To show that μ is a measure we need to show that it is countably additive. For that let $A = \bigsqcup_{n=1}^{\infty} A_n$ with $A, A_n \in \mathcal{A}$.

then,

$$A = \bigcup_{k=1}^{\infty} \left(\bigsqcup_{n=1}^{k} A_n\right) = \bigcup_{k=1}^{\infty} B_k \text{ where } B_k = \bigsqcup_{n=1}^{k} A_n.$$

therefore, as $B_k \uparrow$

$$\mu(A) = \lim_{k\to\infty} \mu(B_k)$$
$$= \lim_{k\to\infty} \mu\left(\bigsqcup_{n=1}^{k} A_n\right)$$
$$= \lim_{k\to\infty} \sum_{n=1}^{k} \mu(A_n)$$
$$= \sum_{n=1}^{\infty} \mu(A_n)$$

Thus, μ is countably additive □.

(*) Theorem 1.18 (Continuity from above)

Let $\mathcal{A}$ be an algebra of subset of a set X and $\mu: \mathcal{A} \to [0,\infty]$ a set function on $\mathcal{A}$ such that $\mu(\emptyset)=0$, $\mu(X)<+\infty$ and μ is finitely additive. Then μ is a measure if and only if μ is continuous from above.

proof

Suppose μ is a measure. Then, μ is countably additive. Let $A\in\mathcal{A}$ such that ~~$A=\bigsqcup_{i=1}^{\infty} A_i$~~ $A=\bigcap_{n=1}^{\infty} A_n$ with $A_n\in\mathcal{A}$ and $A_n \supseteq A_{n+1}$ for all n. Define $B_n = X\setminus A_n$ for all n. Then, $B_n\in\mathcal{A}$ and $B_n\subseteq B_{n+1}$ for all n. By theorem 1.17, we have ~~$\mu(X\setminus A)$~~

$$\mu(X\setminus A) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty}\mu(B_n) = \lim_{n\to\infty}\mu(X\setminus A_n)$$

$$\Rightarrow \mu(X) - \mu(A) = \lim_{n\to\infty}(\mu(X) - \mu(A_n))$$

$\Rightarrow \mu(A) = \lim_{n\to\infty} \mu(A_n)$.

And so, μ is continuous from above.

Conversely, suppose μ is continuous from above. Let $A = \bigsqcup_{i=1}^{\infty} A_i$ where $A, A_i \in \mathcal{A}_0$ for all i. Then $A = \bigcup_{n=1}^{\infty}\left(\bigcup_{i=1}^{n} A_i\right)$ and $X \setminus A = X \setminus \bigcup_{n=1}^{\infty}\left(\bigcup_{i=1}^{n} A_i\right)$

$$= \bigcap_{n=1}^{\infty}\left(X \setminus \bigcup_{i=1}^{n} A_i\right).$$

therefore,

$$\mu(X\setminus A) = \mu\left(\bigcap_{n=1}^{\infty}\left(X \setminus \bigcup_{i=1}^{n} A_i\right)\right) = \lim_{n\to\infty} \mu\left(X \setminus \bigcup_{i=1}^{n} A_i\right)$$

$$\Rightarrow \mu(X) - \mu(A) = \lim_{n\to\infty}\left(\mu(X) - \mu\left(\bigcup_{i=1}^{n} A_i\right)\right)$$

$$\Rightarrow \mu(A) = \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{n} A_i\right)$$

$$\Rightarrow \mu(A) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(A_i).$$

$$\Rightarrow \mu(A) = \sum_{i=1}^{\infty} \mu(A_i).$$

And so, μ is countably additive. Thus, μ is a measure.

Exercises

1. Let X be any countable set, $\mathcal{C}_0 = \{\{x\} \mid x \in X\}$ and $\mathcal{A}(\mathcal{C}_0) = \{A \subseteq X \mid A$ is finite or A^c is finite$\}$ show that the set function $\mu : \mathcal{A}(\mathcal{C}_0) \to [0, \infty]$ define by

$$\mu(A) := \begin{cases} 0, & \text{if } A \text{ is finite} \\ 1, & \text{if } A \text{ is infinite} \end{cases}$$

is finitely additive but not countably additive. Show also that if X is uncountable, then μ is countably additive.

2. Let X be a non-empty set. A collection $\mathcal{U}$ of subsets of X is called an <u>ultrafilter</u> in X if it satisfies the following conditions:

i. $\emptyset \notin \mathcal{U}$

ii. $A \in \mathcal{U}$ and $A \subseteq B \Rightarrow B \in \mathcal{U}$

iii. $A, B \in \mathcal{U} \Rightarrow A \cap B \in \mathcal{U}$

iv. $A \in \mathcal{U}$ or $A^c \in \mathcal{U}$ for all $A \subseteq X$.

a. Let X be a non-empty set and $\mu : P(X) \to [0, \infty]$ be a finitely additive set function such that, for every $A \subseteq X$, $\mu(A) = 0$ or 1. Show that the collection $\mathcal{U} = \{A \subseteq X \mid \mu(A) = 1\}$ is an ultrafilter in X.

b. Let $\mathcal{U}$ be any ultrafilter in X and define $\mu : P(X) \to [0, \infty]$ by $\mu(A) := \begin{cases} 1, & \text{if } A \in \mathcal{U} \\ 0, & \text{if } A \notin \mathcal{U} \end{cases}$

Show that μ is finitely additive.

1.4 The Induced Outer Measure Measurable sets

Definition

Let X be a non-empty set. A set function $\nu : P(X) \to [0, \infty]$ is called an outer measure if

i. $\nu(\phi) = 0$

ii. ν is monotone

iii. ν countably subadditive, $\nu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \nu(A_i)$ whenever $A_i \in P(X)$.

(*) **Theorem 1.19**

Let $\mathcal{A}$ be an algebra of subsets of a set X and $\mu : \mathcal{A} \to [0, \infty]$ be a measure on $\mathcal{A}$. The set function $\mu^* : P(X) \to [0, \infty]$ defined by: For all $E \subseteq X$,

$$\mu^*(E) := \inf\left\{ \sum_{i=1}^{\infty} \mu(A_i) \;\middle|\; A_i \in \mathcal{A} \text{ and } E \subseteq \bigcup_{i=1}^{\infty} A_i \right\}$$

is an outer measure which is an extension of μ. It is called the outer measure induced by μ.

proof

To show that μ^* is an outer measure, we need to show that $\mu^*(\phi) = 0$, μ^* is monotone and μ^* is countably subadditive.

Clearly, from its definition, $\mu^*(\phi) = 0$ and that μ^* is monotone, for if $E \subseteq F$ for $E, F \in P(X)$, then we can choose $A_i, B_j \in \mathcal{A}_0$ such that $E \subseteq \bigcup_{i=1}^{\infty} A_i$ $\subseteq F \subseteq \bigcup_{i=1}^{\infty} B_j$. And so

$$\mu^*(E) \le \sum_{i=1}^{\infty} \mu(A_i) \le \mu^*(F) \le \sum_{j=1}^{\infty} \mu(B_j)$$

$\Rightarrow \mu^*(E) \le \mu^*(F)$.

Now to see that μ^* is countably subadditive let $A = \bigcup_{i=1}^{\infty} A_i$ with $A_i \in P(X)$. If $\mu^*(A) = +\infty$ for some i, then clearly $\mu^*(A) \le +\infty = \mu^*(A_i) \le \sum_{i=1}^{\infty} \mu^*(A_i)$.
If $\mu^*(A_i) < +\infty$ for all i. Then given; given $\varepsilon > 0$, we can find, for each i, $\{A_j^i\}_{j\ge 1}$ such that $A_i \subseteq \bigcup_{j=1}^{\infty} A_j^i$ with each $A_j^i \in \mathcal{A}_0$ and

$$\mu^*(A_i) + \varepsilon/2^i > \sum_{j=1}^{\infty} \mu(A_j^i)$$

But then, $A = \bigcup_{i=1}^{\infty} A_i \subseteq \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} A_j^i$

$$\sum_{i=1}^{\infty} \mu^*(A_i) + \sum_{i=1}^{\infty} \varepsilon/2^i > \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mu(A_j^i) \ge \mu^*(A).$$

Thus,

$$\mu^*(A) < \sum_{i=1}^{\infty} \mu^*(A_i) + \sum_{i=1}^{\infty} \varepsilon/2^i$$

since this holds for every $\varepsilon > 0$, we get

$$\mu^*(A) \le \sum_{i=1}^{\infty} \mu^*(A_i)$$

Therefore, μ^* is countably subadditive.

To prove that μ^* is an extension of μ. Let $A \in \mathcal{A}_0$. Clearly $\mu^*(A) \le \mu(A)$.

To complete the proof we need only to show that $\mu(A) \leq \mu^*(A)$.

If $\mu^*(A) = +\infty$, then we are done.

If $\mu^*(A) < +\infty$, then given any $\varepsilon > 0$, we can choose pairwise disjoint sets $A_n \in \mathcal{A}$ such that $A \subseteq \bigsqcup_{n=1}^{\infty} A_n$ such that

$$\mu^*(A) + \varepsilon \geq \sum_{n=1}^{\infty} \mu(A_n)$$

Since $\left\{ \bigcup_{n=1}^{k} (A_n \cap A) \right\}_{k \geq 1}$ is monotone increasing to A, we use continuity of μ from below to have

$$\sum_{n=1}^{\infty} \mu(A_n) = \lim_{n \to \infty} \sum_{n=1}^{k} \mu(A_n)$$

$$= \lim_{n \to \infty} \mu\left(\bigcup_{n=1}^{k} A_n \right)$$

$$\geq \lim_{n \to \infty} \mu\left(\bigcup_{n=1}^{k} (A_n \cap A) \right)$$

$$= \mu\left(\bigcup_{n=1}^{\infty} (A_n \cap A) \right)$$

$$= \mu(A).$$

(2) Example

Let $X = \mathbb{R}$ and $\mathcal{A} = \{ E \subseteq \mathbb{R} \mid E \text{ is finite or } E^c \text{ is finite} \} = \mathcal{A}(\{\{x\} \mid x \in \mathbb{R}\})$.

Define $\mu : \mathcal{A} \to [0, \infty]$ by, for $E \in \mathcal{A}$

$$\mu(E) = \begin{cases} 0 & \text{if } E \text{ is finite} \\ 1 & \text{if } E^c \text{ is finite} \end{cases}$$

then, clearly $\mu(\emptyset) = 0$.

Let $A \in \mathcal{A}_0$ be such that $A = \bigsqcup_{i=1}^{\infty} A_i$ with $A_i \in \mathcal{A}_0$ for all i, If each A_i is finite, then $A = \bigsqcup_{i=0}^{\infty} A_i$ must be finite, since $\mathbb{R}$ is uncountable. And so,

$$\mu(O) = 0 = \sum_{i=1}^{\infty} \mu(A_i)$$

If A_{i_0} is not finite for some i_0, then $A_{i_0}^c$ is finite. And so, since

$$A_{i_0} \cap A_j = \emptyset \text{ for all } j,$$

we have for all j, $A_j \subset A_{i_0}^c$.

Hence, $\mu(A_j) = 0$ for all $j \neq i_0$.

Also, $A^c \subset A_{i_0}^c$. Thus, $\mu(A) = 1$

Therefore,

$$\sum_{i=1}^{\infty} \mu(A_i) = 1 = \mu(A).$$

Hence μ is countably additive and so a measure on $\mathcal{A}_0$.

(*) Now, the outer measure μ^* induced by μ on $P(\mathbb{R})$ is monotone and countably subadditive is μ^* a measure on $P(\mathbb{R})$. To see this we need only show that μ^* is finitely additive.

If $A \subseteq \mathbb{R}$ is countable, then $A = \bigcup_{i=1}^{\infty} \{x_i\}$ and so

$$\mu^*(A) = \sum_{i=1}^{\infty} \mu(\{x_i\}) = 0,$$

If $A \subseteq \mathbb{R}$ is uncountable, then clearly,
$\mu^*(A) \leq \mu^*(\mathbb{R}) = 1$.
Also, if $A_i \in \mathcal{A}_0$, $i \geq 1$ and such that
$A \subseteq \bigcup_{i=1}^{\infty} A_i$, then A being countable,
implies that, for some i_0, A_{i_0} is uncountable,
since $A_{i_0} \in \mathcal{A}_0$, we have $A_{i_0}^c$ is finite. Thus

$$\sum_{i \geq 1}^{\infty} \mu(A_i) \geq \mu(A_{i_0}) \geq 1.$$

Thus, $\mu^*(A) \geq 1$.
Hence, $\mu^*(A) = 1$ iff A is uncountable.
Now, $\mathbb{R} = (-\infty, 0] \sqcup (0, \infty)$ and

$$\mu^*(\mathbb{R}) = 1 < 2 = \mu^*((-\infty, 0]) + \mu^*((0, \infty))$$

Hence, μ^* is not finitely additive and so not countably additive on $P(\mathbb{R})$

<u>Remark</u>

~~The outer measure μ^* on $P(\mathbb{R})$ is a measure on the collection $\mathcal{C}_0 = \{$~~
~~instead of using the algebra $\mathcal{A}_0 = \{E \subseteq \mathbb{R} \mid E$ or E^c is finite$\}$~~

Instead of using $P(\mathbb{R})$ for μ^* if we use
$\mathcal{C}_0 = \{E \subseteq \mathbb{R} \mid E$ or E^c is countable$\}$.
($\mu^* : \mathcal{C}_0 \to [0, \infty]$), then μ^* is a measure on $\mathcal{C}_0$.
The example shows that μ^* may not be finitely additive on $P(X)$.

Suppose $\mathcal{C} \subseteq P(X)$ with $\mathcal{A} \subseteq \mathcal{C}$ such that μ^* is finitely additive on $\mathcal{C}$. Then we have

Lemma 1.20

~~For all $Y \subseteq X$ and $E \in \mathcal{C}$, we have~~

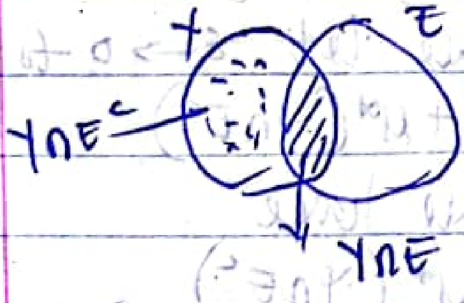
~~$$\mu^*(Y) = \mu^*(Y \cap E) + \mu^*(Y \cap E^c).$$~~

(*) **Lemma 1.20**

Let $\mathcal{A}$ be an algebra on X with a measure $\mu : \mathcal{A} \to [0, \infty]$ and let $\mathcal{C} \subseteq P(X)$ on which μ^* is countably additive. If $E \in \mathcal{C}$, then, for all $Y \subseteq X$, we have

$$\mu^*(Y) = \mu^*(Y \cap E) + \mu^*(Y \cap E^c).$$

proof



Let $Y \subseteq X$ and $E \in \mathcal{C}$. Then, by countable subadditivity of μ^*, we have

$$\mu^*(Y) \leq \mu^*(Y \cap E) + \mu^*(Y \cap E^c)$$

If $\mu^*(Y) = +\infty$, then

$$\mu^*(Y) \geq \mu^*(Y \cap E) + \mu^*(Y \cap E^c)$$

If $\mu^*(Y) < +\infty$, then given any $\varepsilon > 0$, we can choose subsets $A_1, A_2, \ldots \in \mathcal{A}_0$ such that

$$Y \subseteq \bigcup_{i=1}^{\infty} A_i$$

and, definition of μ^*

$$\mu^*(Y) + \varepsilon \geq \sum_{i=1}^{\infty} \mu(A_i) = \sum_{i=1}^{\infty} \mu^*(A_i)$$

Now, since μ^* is countably additive and in particular finitely additive on $\mathcal{T}_0$, we have, for each i,

$$\mu^*(A_i) = \mu^*(A_i \cap E) + \mu^*(A_i \cap E^c)$$ And so,

$$\mu^*(Y) + \varepsilon \geq \sum_{i=1}^{\infty} [\mu^*(A_i \cap E) + \mu^*(A_i \cap E^c)$$

$$= \sum_{i=1}^{\infty} \mu^*(A_i \cap E) + \sum_{i=1}^{\infty} \mu^*(A_i \cap E^c)$$

$$\geq \mu^*\left(\bigcup_{i=1}^{\infty} A_i \cap E\right) + \mu^*\left(\bigcup_{i=1}^{\infty} A_i \cap E^c\right)$$

$$= \mu^*(Y \cap E) + \mu^*(Y \cap E^c)$$

since ε is arbitrary we let $\varepsilon \to 0$ to have

$$\mu^*(Y) \geq \mu^*(Y \cap E) + \mu^*(Y \cap E^c).$$

Hence, in all cases, we have

$$\mu^*(Y) = \mu^*(Y \cap E) + \mu^*(Y \cap E^c)$$ $\square$.

Definition.

Let $\mu : \mathcal{A}_0 \to [0, \infty]$ be a measure on the algebra $\mathcal{A}_0$ of subsets of X. Then a subset $E \subseteq X$

is said to be μ^* - measurable iff for all $Y \subseteq X$,

$$\mu^*(Y) = \mu^*(Y \cap E) + \mu^*(Y \cap E^c)$$

The next theorem shows that for a subset $E \subseteq X$ to be μ^* measurable it suffices to only satisfy a weaker condition than the one in the definition.

Theorem 1.21

Let $E \subseteq X$. Then, the following are equivalent:

i. E is μ^* - measurable

ii. E^c is μ^* - measurable

iii. for all $Y \subseteq X$

$$\mu^*(Y) \geq \mu^*(Y \cap E) + \mu^*(Y \cap E^c)$$

iv For all $Y \subseteq X$ with $\mu^*(Y) < +\infty$,

$$\mu^*(Y) \geq \mu^*(Y \cap E) + \mu^*(Y \cap E^c)$$

v. For all $A \in \mathcal{A}$,

$$\mu(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

proof:

(i) $\Leftrightarrow$ (ii) is clear by the symmetry of equa ①

(ii) $\Leftrightarrow$ (iii) is also clear by countably subadditive of μ^*

(iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) as obvious

(v) $\Rightarrow$ (iii) Let $Y \subseteq X$. If $\mu^*(Y) = +\infty$, then clearly $\mu^*(Y) \geq \mu^*(Y \cap E) + \mu^*(Y \cap E^c)$

If $\mu^*(Y) < +\infty$, then for any given $\varepsilon > 0$, we choose $A_1, A_2, \ldots \in \mathcal{A}$ such that

$Y \subseteq \bigcup_{i=1}^{\infty} A_i$ and

$$\mu^*(Y) + \varepsilon > \sum_{i=1}^{\infty} \mu(A_i)$$

$$\geq \sum_{i=1}^{\infty} [\mu^*(A_i \cap E) + \mu^*(A_i \cap E^c)]$$

$$= \sum_{i=1}^{\infty} \mu^*(A_i \cap E) + \sum_{i=1}^{\infty} \mu^*(A_i \cap E^c)$$

$$\geq \mu^*(Y \cap E) + \mu^*(Y \cap E^c)$$

Since $\varepsilon > 0$ is arbitrary, we hence

$\mu^*(Y) \geq \mu^*(Y \cap E) + \mu^*(Y \cap E^c)$ for all $Y \subseteq X$. ▫

Test

Exercise

If $\mu(X) < +\infty$, then $E \subseteq X$ is μ^*-measurable if and only if

$$\mu(X) = \mu^*(E) + \mu^*(E^c).$$

Given a measure $\mu : \mathcal{A} \to [0, \infty]$ on the algebra of subsets of a set X, we denote by $\mathcal{C}_0$ be collection of all μ^*-measurable subsets of X. i.e.

$$\mathcal{C}_0^* = \{E \subseteq X \mid E \text{ is } \mu^*\text{-measurable}\}.$$

$\mathcal{N}$ is μ^*-null subset.

(*) **Theorem 1.22**

The collection $\mathcal{C}_c^*$ has the following properties.

i. $A_c \subseteq \mathcal{C}_c^*$

ii. $\mathcal{C}_c^*$ is an algebra of subsets of X and $\mu^*|_{\mathcal{C}_c^*}$ (the restriction of μ^* on $\mathcal{C}_c^*$) is finitely additive.

iii. If $A_1, A_2, A_3, \dots \in \mathcal{C}_c^*$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{C}_c$ and $\mu^*|_{\mathcal{C}_c^*}$ is countably additive

iv. $\mathcal{N} := \{E \subseteq X \mid \mu^*(E) = 0\} \subseteq \mathcal{C}_c^*$

proof

i. Let $Y \subseteq X$ be any subset of X and $E \in A_c$. Choose $A_i \in A_c$, $i \geq 1$ such that $A_i \cap A_j = \phi$. $i \neq j$ and $E = \bigcup_{i=1}^{\infty} A_i$. Then, by countable additivity of μ, we have

$$\sum_{i=1}^{\infty} \mu(A_i) = \sum_{i=1}^{\infty} \mu(A_i \cap E) + \sum_{i=1}^{\infty} \mu(A_i \cap E^c)$$

$$\geq \mu^*(Y \cap E) + \mu^*(Y \cap E^c)$$

Since this holds for any pairwise disjoint covering $\{A_i\}_{i \geq 1}$ of Y by elements of A, we have

$$\mu^*(Y) \geq \mu^*(Y \cap E) + \mu^*(Y \cap E^c)$$

Hence, $E \in \mathcal{C}_c^*$ so that $A_c \subseteq \mathcal{C}_c^*$

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ii. Clearly $\emptyset \in \mathcal{C}_e^*$ and if $A \in \mathcal{C}_e^*$, then $A^c \in \mathcal{C}_e^*$ by theorem 1.21. Finally, let $A_1, A_2 \in \mathcal{C}_e$, then for all $Y \subseteq X$ with $\mu^*(Y) < +\infty$,

$$\mu^*(Y) = \mu^*(Y \cap A_1) + \mu^*(Y \cap A_1^c)$$

Replacing Y by $Y \cap (A_1 \cup A_2)$ in this last equation we have

$$\mu^*(Y \cap (A_1 \cup A_2)) = \mu^*((Y \cap (A_1 \cup A_2)) \cap A_1) + \mu^*((Y \cap (A_1 \cup A_2)) \cap A_1^c)$$

i.e.

$$\mu^*(Y \cap (A_1 \cup A_2)) = \mu^*(Y \cap A_1) + \mu^*(Y \cap A_2 \cap A_1^c)$$

Also, since $A_2 \in \mathcal{C}_e^*$, we have ①

$$\mu^*(Y \cap A_1^c) = \mu^*(Y \cap A_1^c \cap A_2) + \mu^*(Y \cap A_1^c \cap A_2^c) \quad \text{②}$$

From ① & ② we have

$$\mu^*(Y \cap (A_1 \cup A_2)) + \mu^*(Y \cap (A_1 \cup A_2)^c)$$
$$= \mu^*(Y \cap A_1) + \mu^*(Y \cap A_1^c)$$
$$= \mu^*(Y)$$

Thus, $A_1 \cup A_2 \in \mathcal{C}_e^*$

In particular, if $A_1, A_2 \in \mathcal{C}_e^*$ are such that $A_1 \cap A_2 = \emptyset$, then, using $Y = A_1 \sqcup A_2$ in equa ① we have $\mu^*(A_1 \sqcup A_2) = \mu^*(A_1) + \mu^*(A_2)$ showing that $\mu^*|_{\mathcal{C}_e^*}$ is finitely additive.

iii. Let $A_i \in \mathcal{C}^*$, $i \geq 1$, and let $A := \bigcup_{i=1}^{\infty} A_i$.
Since, from (ii), $\mathcal{C}^*$ is an algebra, there is no loss of generality in assuming that $A_i \cap A_j = \emptyset$ for $i \neq j$.

Now, for every $Y \subseteq X$, we have

$$\mu^*(Y) \geq \mu^*(Y \cap A_1) + \mu^*(Y \cap A_1^c)$$

$$\geq \mu^*(Y \cap A_1) + \mu^*(Y \cap A_1^c \cap A_2) + \mu^*(Y \cap A_1^c \cap A_2^c)$$

$$\geq \mu^*(Y \cap A_1) + \mu^*(Y \cap A_2) + \mu^*(Y \cap A_1^c \cap A_2^c \cap A_3) + \mu^*(Y \cap A_1^c \cap A_2^c \cap A_3^c)$$

Thus after n-steps we have

$$\mu^*(Y) \geq \sum_{i=1}^{n} \mu^*(Y \cap A_i) + \mu^*(Y \cap (\bigcap_{i=1}^{n} A_i^c))$$

$$\geq \sum_{i=1}^{n} \mu^*(Y \cap A_i) + \mu^*(Y \cap (\bigcup_{i=1}^{n} A_i)^c)$$

Since this equality is for all $n \geq 1$ we have that

$$\mu^*(Y) \geq \sum_{i=1}^{\infty} \mu^*(Y \cap A_i) + \mu^*(Y \cap (\bigcup_{i=1}^{\infty} A_i)^c) \quad \text{— (3)}$$

But μ^* is countably sub-additive, thus we have that for all $Y \subseteq X$,

$$\mu^*(Y) \geq \mu^*(Y \cap (\bigcup_{i=1}^{\infty} A_i)) + \mu^*(Y \cap (\bigcup_{i=1}^{\infty} A_i)^c)$$

Thus $\bigcup_{i=1}^{\infty} A_i$ is μ^* measurable and so $\bigcup_{i=1}^{\infty} A_i \in \mathcal{C}$.
In particular, if we choose $Y = \bigcup_{i=1}^{\infty} A_i$ in equation (3) we have

$$\mu^*(\bigcup_{i=1}^{\infty} A_i) \geq \sum_{i=1}^{\infty} \mu^*(A_i) \quad \text{— (4)}$$

By countable sub-additivity, we have

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \le \sum_{i=1}^{\infty} \mu^*(A_i)$$

from (4) & (3) we have

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu^*(A_i)$$

and so $\mu^*|_{\mathcal{C}^*}$ is countably additive.

iv. Let $E \subseteq X$ with $\mu^*(E) = 0$. Then for all $Y \subseteq X$,

$$\mu^*(Y) \ge \mu^*(Y \cap E^c) + 0$$
$$= \mu^*(Y \cap E^c) + \mu^*(Y \cap E)$$

so that $E \in \mathcal{C}^*$.

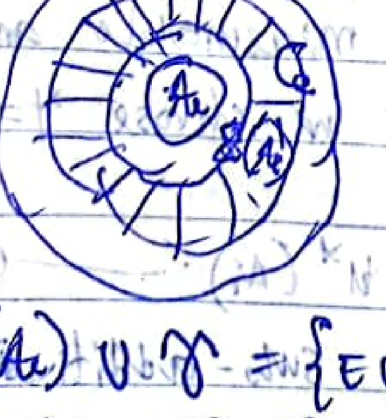
<u>Corollary 1.23</u>

The class $\mathcal{C}^*$ of all μ^*-measurable subsets of X is a σ-algebra containing $\mathcal{A}_c$.

<u>Remark</u>

i. $\mathcal{A}_c \subseteq \sigma(\mathcal{A}_c) \subseteq \mathcal{C}^* \subseteq P(X)$

ii. $\gamma := \{E \subseteq X \mid \mu^*(x) = 0\}$.



$\mathcal{C}_c = \sigma(\mathcal{A}_c) \cup \gamma = \{E \cup N \mid E \in \sigma(\mathcal{A}_c) \text{ and } N \in \gamma\}$

μ^* is a measure on $\sigma(\mathcal{A}_c)$

Definition.

Let X be a non-empty set, $\mathcal{E}$ be a σ-algebra of subsets of X. Then the pair $(X, \mathcal{E})$ is called a measurable space, elements of $\mathcal{E}$ are called measurable subsets. If $\mu: \mathcal{E} \to [0, \infty]$ is a measure on $\mathcal{E}$, then the triple $(X, \mathcal{E}, \mu)$ is called a measure space.

If $\mathcal{A}$ is an algebra of subsets of X and $\mu: \mathcal{A} \to [0, \infty]$ a measure on $\mathcal{A}$, we have a measure spaces $(X, \sigma(\mathcal{A}), \mu^*)$ and $(X, \mathcal{E}, \mu^*)$. The measure space $(X, \mathcal{E}, \mu^*)$ has the property that; If $E \subseteq X$ with $\mu^*(E) = 0$, then $E = \mathcal{E}^*$. This property is called the completeness of the measure space and the space is said to be complete.

The space $(X, \sigma(\mathcal{A}), \mu^*)$ is not in general complete, but we have the following:

Theorem 1.24

Every measure space $(X, \mathcal{E}, \mu)$ can be completed

Definition.

Let X be a non-empty set, $\mathcal{E}$ be a σ-algebra of subsets of X. Then the pair $(X, \mathcal{E})$ is called a measurable space, elements of $\mathcal{E}$ are called measurable subsets. If $\mu: \mathcal{E} \to [0, \infty]$ is a measure on $\mathcal{E}$, then the triple $(X, \mathcal{E}, \mu)$ is called a measure space.

If $\mathcal{A}$ is an algebra of subsets of X and $\mu: \mathcal{A} \to [0, \infty]$ a measure on $\mathcal{A}$, we have a measure spaces $(X, \sigma(\mathcal{A}), \mu^*)$ and $(X, \mathcal{C}, \mu^*)$. The measure space $(X, \mathcal{C}, \mu^*)$ has the property that; If $E \subseteq X$ with $\mu^*(E) = 0$, then $E \in \mathcal{C}$. This property is called the completeness of the measure space and the space is said to be complete.

The space $(X, \sigma(\mathcal{A}), \mu^*)$ is not in general complete, but we have the following:

Theorem 1.24

Every measure space $(X, \mathcal{E}, \mu)$ can be completed

Exercise

Let $\mathcal{A}$ be an algebra of subset of a non-empty set X and $\mu: \mathcal{A} \to [0, \infty]$ be a measure on $\mathcal{A}$. Let $E \subseteq X$. A subset $F \in S(\mathcal{A})$ is called a measurable cover of E if

i. $E \subseteq F$

ii. $\mu^*(E) = \mu^*(F)$

iii. $\mu^*(F \setminus E) = 0$

A subset $K \in S(\mathcal{A})$ is called a measurable kernel of E if

i. $K \subseteq E$

ii. For $A \subseteq E \setminus K$, $\mu^*(A) = 0$ ~~that~~

prove ~~that~~ the existence of measurable cover and hence deduce that of measurable kernel.

2. Lebesgue measure on $\mathbb{R}$.

We now specialize to the case when $X = \mathbb{R}$, $A_l = A_l(\mathcal{I})$, the algebra generated by all intervals in $\mathbb{R}$, and $\mu = \lambda$, the length function on $A_l(\mathcal{I})$.

Definition

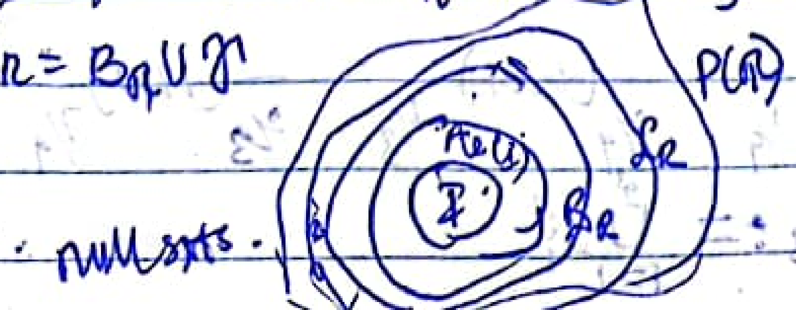
The outer measure λ^* induced by the length function λ on $A_l(\mathcal{I})$ is called the Lebesgue outer measure and can be described by, for each $E \subseteq \mathbb{R}$,

$$\lambda^*(E) := \inf \left\{ \sum_{i=1}^{\infty} \lambda(I_i) \mid I_i \in \mathcal{I}, \ I_i \cap I_j = \emptyset, \ i \neq j \text{ and } E \subseteq \bigcup_{i=1}^{\infty} I_i \right\}$$

The σ-algebra of all λ^*-measurable subsets of $\mathbb{R}$ is called the σ-algebra of Lebesgue-measurable sets and is denoted by $\mathcal{L}_{\mathbb{R}}$ (or simply $\mathcal{L}$).

The σ-algebra $\delta(A_l(\mathcal{I})) = \delta(\mathcal{I})$ is called the σ-algebra of Borel subsets of $\mathbb{R}$ and is denoted by $\mathcal{B}_{\mathbb{R}}$ (or simply $\mathcal{B}$). The outer measure space $(\mathbb{R}, \mathcal{L}_{\mathbb{R}}, \lambda^*)$ is called the Lebesgue measure space and λ^* is called the Lebesgue measure & it is usually denoted by λ itself.

$\mathcal{L}_{\mathbb{R}} = \mathcal{B}_{\mathbb{R}} \cup \mathcal{N}$



$\cdot B_{\mathbb{R}} \subseteq \mathcal{L}_{\mathbb{R}} \subseteq P(\mathbb{R})$

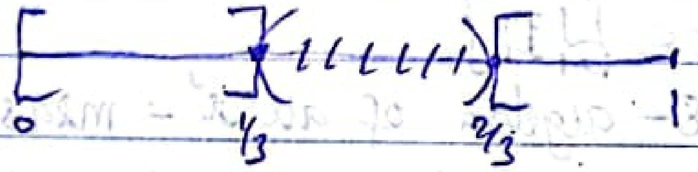
$$A = \bigcup_{i=1}^{\infty} \{x_i\}$$

$$\lambda(A) = \sum_{i=1}^{\infty} \lambda(\{x_i\}) = 0$$

Example (Cantor's ternary set)

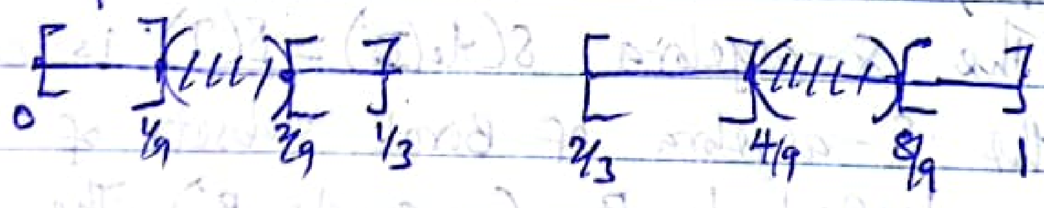
Consider the interval $[0,1] = A_0$. We construct a set as follows:

Step 1:



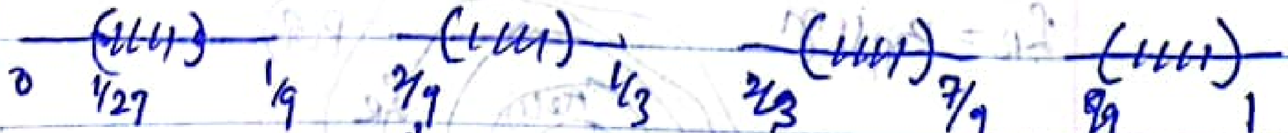
$$A_1 = [0, 1/3] \cup [2/3, 1] = \bigcup_{i=1}^{2} I_i^1$$

Step 2



$$A_2 = \underset{I_1^2}{[0, 1/9]} \cup \underset{I_2^2}{[2/9, 1/3]} \cup \underset{I_3^2}{[2/3, 7/9]} \cup \underset{I_4^2}{[8/9, 1]} = \bigcup_{i=1}^{2^2} I_i^2$$

Step 3



$$A_3 := \bigcup_{i=1}^{2^3} I_i^3$$

$\vdots$

step n:

$$A_n := \bigcup_{i=1}^{2^n} I_i^n$$

The cantor's ternary set is defined to be $C := \bigcap_{n=1}^{\infty} A_n$.

We observe that:

① the cantor's ternary set $C \neq \emptyset$. Infact the end points of the open intervals at every stage are members of C $(0, 1, 1/3, 2/3, 1/9, 2/3, 1/9, 2/9, 7/9, \ldots)$

② C is uncountable. To see this, define a map $f: [0,1] \to C$ which is 1-1. For this map, let $x \in [0,1]$ and consider its binary expression given by $x = 0.a_1a_2a_3\ldots$ with each $a_n = 0$ or 1. Then, construct a point $y \in [0,1]$ with ternary expression $y = 0.b_1b_2b_3\ldots$ where, for all n, $b_n = 2a_n$. Then $y \in C$. Hence, we define $f: [0,1] \to C$ by $f(x) = y$. Then f is 1-1, for if $x_1 = 0.a_1^1a_2^1a_3^1\ldots$ and $x_2 = 0.a_1^2a_2^2a_3^2\ldots$ such that $x_1 \neq x_2$, then $\exists n_0 \ni a_{n_0}^1 \neq a_{n_0}^2 \Rightarrow 2a_{n_0}^1 \neq 2a_{n_0}^2 \Rightarrow b_{n_0}^1 \neq b_{n_0}^2$ $\Rightarrow y_1 \neq y_2 \Rightarrow f(x_1) \neq f(x_2)$. Hence,

$\#(C) = \#([0,1]) = \#(\mathbb{R}) = c$, the continuum.

<u>Note that</u>

$$C = \bigcap_{n=1}^{\infty} A_n \Rightarrow C \subseteq A_n = \bigcup_{i=1}^{2^n} I_i^n$$

for each n. But $\ell(I_i^n) = \frac{1}{3^{2^n-1}}$ and so

$$\sum_{i=1}^{2^n} \lambda(I_i^n) = \frac{2^n}{3^n} - 1 \xrightarrow[n\to\infty]{} 0$$

Thus, $\lambda^*(C) = 0$ so that C is λ^*-n set. Hence $C \in \mathcal{L}_R$.

In fact, for all $E \subseteq C$, $\lambda^*(E) = 0$ as $E \in \mathcal{L}_R$. Therefore

$P(C) \subseteq \mathcal{L}_R \subseteq P(R)$

$\Rightarrow 2^{2^{\aleph_0}} \subseteq \mathcal{L}_R \subseteq 2^{2^{\aleph_0}} \Rightarrow \#(\mathcal{L}_R) = 2^{2^{\aleph_0}}$

Example (construction of a non-measurable)

Define a relation $\sim$ on $[0,1]$ by $x \sim y$ iff $x - y$ is rational. Let $\{E_\alpha\}_{\alpha \in I}$ denote the set of all equivalence classes of elements of $[0,1]$. Using axioms of choice, we choose exactly one element $x_\alpha \in E_\alpha$ for each $\alpha \in I$ and construct the set $E := \{x_\alpha \mid \alpha \in I\}$. Let $r_1, r_2, r_3, \ldots$ be an enumeration of rational numbers in $[-1,1]$. Define $E_n := E + r_n$, $n \geq 1$. It is easy to check that $E_i \cap E_j = \phi$ for $i \neq j$ and that $E_n \subseteq [-1,2]$ $\forall n$.

If $x \in [0,1]$, then $x \in E_\alpha$ for some $\alpha \in I$ and hence $x \sim x_\alpha$. But $x - x_\alpha$ is a rational in $[-1,1]$.

Hence, $x \in E_n$ for some n. Thus,

$[0,1] \subseteq \bigcup_{n=1}^{\infty} E_n \subseteq [-1,2]$.

suppose $E \in \mathcal{L}_{\mathbb{R}}$, then by translation invariance of λ, $E_n \in \mathcal{L}_{\mathbb{R}}$ and $\lambda(E_n) = \lambda(E)$. If $\lambda(E) > 0$ then $\lambda(E_n) > 0$ for all n. But then by countable additivity of λ,

$$\infty = \sum_{n=1}^{\infty} \lambda(E_n) = \lambda\left(\bigcup_{n=1}^{\infty} E_n\right) \leq 3$$

which is impossible

If $\lambda(E) = 0$, then $\lambda(E_n) = 0$ $\forall n$. And Hence,

$$1 = \lambda([0,1]) \leq \sum_{n=1}^{\infty} \lambda(E_n) = 0$$

which is again impossible. Hence, E is not Lebesgue measurable.

Theorem 2.1 (Translation invarience).

For each $x \in \mathbb{R}$ and $E \subseteq \mathbb{R}$

i. $E \in \mathcal{L}_{\mathbb{R}} \Rightarrow E + x \in \mathcal{L}_{\mathbb{R}}$

ii. $E \in \mathcal{B}_{\mathbb{R}} \Rightarrow E + x \in \mathcal{B}_{\mathbb{R}}$

proof

i- If $E \in \mathcal{L}_{\mathbb{R}}$, then for all $Y \subseteq \mathbb{R}$,

$$\lambda(Y) = \lambda(Y \cap E) + \lambda(Y \cap E^c)$$

Now,

$$E \subseteq \bigcup_{i=1}^{\infty} I_i \Leftrightarrow E + x \subseteq \bigcup_{i=1}^{\infty} (I_i + x)$$

and $\lambda(E) = \lambda(E+x)$. Thus

$$\lambda(Y) = \lambda(Y \cap E) + \lambda(Y \cap E^c)$$
$$= \lambda((Y \cap E) + x) + \lambda((Y \cap E^c) + x)$$
$$= \lambda((Y+x) \cap (E+x)) + \lambda((Y+x) \cap (E^c+x))$$

Replacing Y by $Y-x$ we have,

$$\lambda(Y-x) = \lambda(Y \cap (E+x)) + \lambda(Y \cap (E^c+x))$$
$$\lambda(Y) = \lambda(Y \cap (E+x)) + \lambda(Y \cap (E+x)^c)$$

Thus, $E+x$ is is λ-measurable and so $E+x \in \mathcal{L}_R$

ii consider the map $\mathbb{R} \to \mathbb{R}$ given by $y \mapsto y+x$. This map is a homeomorphism. Let

$$\mathcal{T}_0 := \{E \in \mathcal{B}_{\mathbb{R}} \mid E+x \in \mathcal{B}_{\mathbb{R}}\}$$

Then each (open) interval in $\mathbb{R}$ is a member of $\mathcal{T}_0$ and $\mathcal{T}_0$ is a σ-algebra. Thus,

$$\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{T}_0 \subseteq \mathcal{B}_{\mathbb{R}} \Rightarrow \mathcal{T}_0 = \mathcal{B}_{\mathbb{R}} \ \square$$

Theorem 2.2

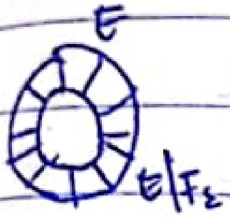
For any set $E \subseteq \mathbb{R}$, the following are equivalent

~~1. $\forall \varepsilon > 0$, $\exists$ an~~

i. $E \in \mathcal{L}_R$ i.e E is a Lebesgue measurable set

ii. $\forall \varepsilon > 0$, $\exists$ an open set G_ε in $\mathbb{R}$ such that $E \subseteq G_\varepsilon$ and $\lambda(G_\varepsilon \setminus E) < \varepsilon$

iii. $\exists$ a G_δ-set (a countable intersection of open sets) G such that $E \subseteq G$ and $\mu(G \setminus E) = 0$



iv. $\forall \varepsilon > 0$ $\exists$ a closed set F_ε in $\mathbb{R}$ such that $F_\varepsilon \subseteq E$ and $\mu(E \setminus F_\varepsilon) < \varepsilon$.

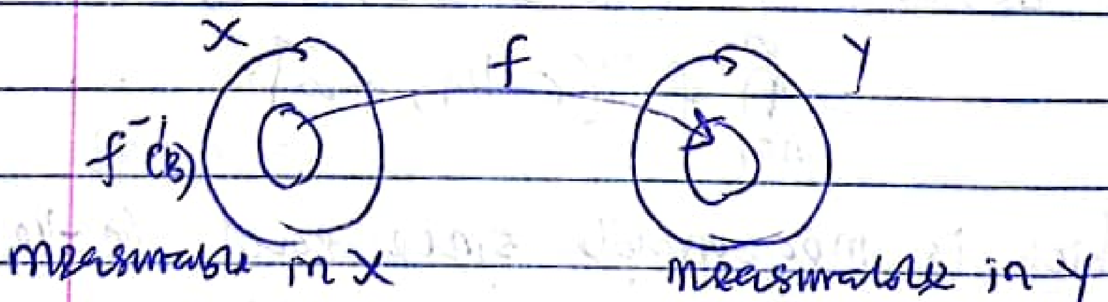
v. $\exists$ F_σ-set (a countable union of closed sets) F such that $F \subseteq E$ and $\mu(E \setminus F) = 0$

3. Measurable Functions

Definition

Let (X, Σ_1) and (Y, Σ_2) be measurable spaces. A map $f: X \to Y$ is said to be measurable if, for all subsets $B \subseteq Y$,

$$B \in \Sigma_2 \Rightarrow f^{-1}(B) \in \Sigma_1$$



Theorem 3.1

Let X be a measurable space and $f: X \to \mathbb{R}^*$. The following are equivalent:

i. f is measurable

ii. For interval I in $\mathbb{R}^*$, $f^{-1}(I)$ is measurable

iii. $f^{-1}(c, +\infty]$ is measurable $\forall c \in \mathbb{R}$

iv. $f^{-1}[c, +\infty]$ is measurable $\forall c \in \mathbb{R}$

v. $f^{-1}[-\infty, c)$ is measurable $\forall c \in \mathbb{R}$

vi. $f^{-1}[-\infty, c]$ is measurable $\forall c \in \mathbb{R}$

vii. $f^{-1}(\{+\infty\})$, $f^{-1}(\{-\infty\})$ and $f^{-1}(E)$ are measurable for all $E \in \mathcal{B}_{\mathbb{R}}$

proof

(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is clear

(ii) $\Leftrightarrow$ (iv): $f^{-1}[c, +\infty] = \{x \in X \mid f(x) \in [c, +\infty]\}$

But then, we note that

$$[c, +\infty] = \bigcap_{n=1}^{\infty} (c - 1/n, +\infty]$$

Therefore,

$$f^{-1}[c, +\infty] = f^{-1}\left(\bigcap_{n=1}^{\infty} (c - 1/n, +\infty]\right)$$

$$= \bigcap_{n=1}^{\infty} f^{-1}(c - 1/n, +\infty]$$

which is measurable since each $(c - 1/n, +\infty]$ is assumed to be measurable.

(iv) $\Leftrightarrow$ (v): since $[-\infty, c) = \mathbb{R}^* \setminus [c, +\infty]$ we have $f^{-1}[-\infty, c) = f^{-1}(\mathbb{R}^* \setminus [c, +\infty])$

$$= f^{-1}(\mathbb{R}^*) \setminus f^{-1}[c, +\infty]$$

$\therefore f^{-1}[-\infty, c) = X \setminus f^{-1}[c, +\infty]$

which is measurable

(v) ⇔ (vi): since $[-\infty, c] = \bigcap_{n=1}^{\infty}[-\infty, c+1/n)$

we have; $[-\infty, c] = \bigcup_{n=1}^{\infty}[-\infty, c-1/n]$

$f^{-1}[-\infty, c] = \bigcap_{n=1}^{\infty} f^{-1}[-\infty, c+1/n)$

which is measurable.

(vi) ⇒ (vii): Note, $\{-\infty\} = \bigcap_{n=1}^{\infty}[-\infty, -n)$ and $\{+\infty\} = \bigcap_{n=1}^{\infty}(n, +\infty]$, And so,

$f^{-1}\{-\infty\} = \bigcap_{n=1}^{\infty} f^{-1}[-\infty, -n)$ and $f^{-1}\{+\infty\} = \bigcap_{n=1}^{\infty}(n, +\infty]$

are measurable by (iii) & (v).

Define $\mathcal{B} := \{E \in \mathcal{B}_{\mathbb{R}} \mid f^{-1}(E)$ is measurable.

Then every interval I of $\mathbb{R}$ is in $\mathcal{B}$ and $\mathcal{B}$ is a σ-algebra.

Thus, $\mathcal{B} = \mathcal{B}_{\mathbb{R}}$

(vii) ⇒ (i). (Exercise).

Example ① Let X be a non-empty set and $A \subseteq X$. Define the <u>characteristic function</u> $\chi_A : X \to \{0,1\}$ by

$$\chi(x) = \begin{cases} 0, & \text{if } x \notin A \\ 1, & \text{if } x \in A \end{cases}$$

If (X, Σ) is a measurable space, then $\chi : X \to \mathbb{R}^*$ is measurable iff A is measurable

proof

If χ_A is measurable, then

$\chi_A^{-1}(\{1\}) = A$ is measurable

Conversely, if A is measurable, then for any interval $I \subseteq \mathbb{R}^*$,

$$\chi_A^{-1}(I) = \begin{cases} \phi, & \text{if } 0,1 \notin I \\ A, & \text{if } 0 \notin I, 1 \in I \\ A^c, & \text{if } 0 \in I, 1 \notin I \\ X, & \text{if } 0,1 \in I \end{cases}$$

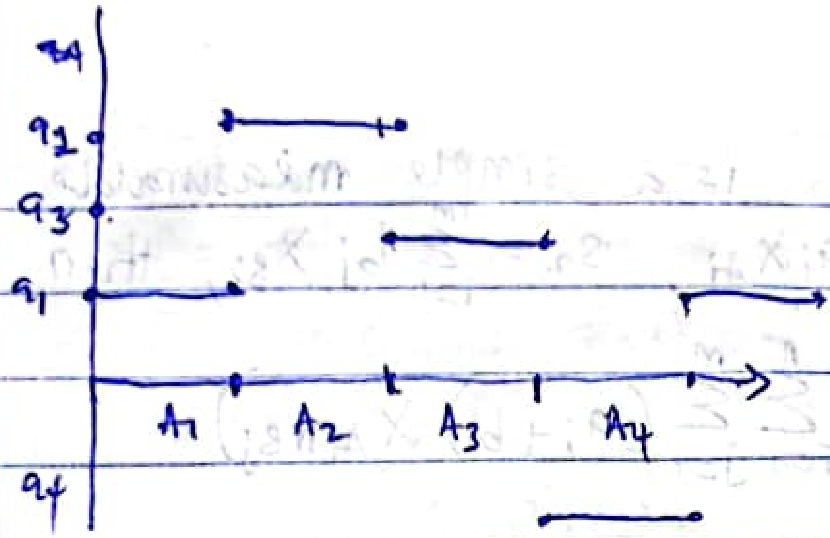
Thus, in all cases $\chi_A^{-1}(I)$ is measurable and so χ_A is measurable.

(*) Definition (simple function)

A function $s : X \to \mathbb{R}^*$ is called simple function if there exist $a_1, a_2, \ldots, a_n \in X$ and subsets $A_i \subseteq X$, $i = 1, 2, \ldots, n$ such that $A_i \cap A_j = \phi$ for $i \neq j$, and $\bigcup_{i=1}^{n} A_i = X$ and

$$s(x) := \sum_{i=1}^{n} a_i \chi_{A_i}(x)$$

If each of the $a_1, a_2, \ldots, a_n$ are non-negative then the function s is said to be non-negative.



Example (2).

A non-negative simple function $s: X \to \mathbb{R}^*$ given by $s = \sum_{i=1}^{n} a_i \chi_{A_i}$ is measurable iff each A_i is measurable

proof

If $s = \sum_{i=1}^{n} a_i \chi_{A_i}$ is measurable, then for each $i = 1, 2, \ldots, n$,

$s^{-1}(\{a_i\}) = A_i$ is measurable.

Conversely, If each A_i is measurable, then for any interval $I \subseteq \mathbb{R}^*$

$s^{-1}(I) = \bigcup_{a_i \in I} A_i$ which is measurable, then s is measurable.

Theorem 3.2

Let s, s_1, s_2 be (non-negative) simple measurable functions on X and $\alpha \in \mathbb{R}$. Then

i. Every constant function is simple

ii. αs is a simple measurable function.

iii. $s_1 + s_2$ is a simple measurable function.
($s_1 = \sum_{i=1}^{n} a_i \chi_{A_i}$, $s_2 = \sum_{i=1}^{m} b_j \chi_{B_i}$, then

$$s_1 + s_2 = \sum_{i=1}^{n}\sum_{j=1}^{m} (a_i + b_i) \chi_{A_i \cap B_j})$$

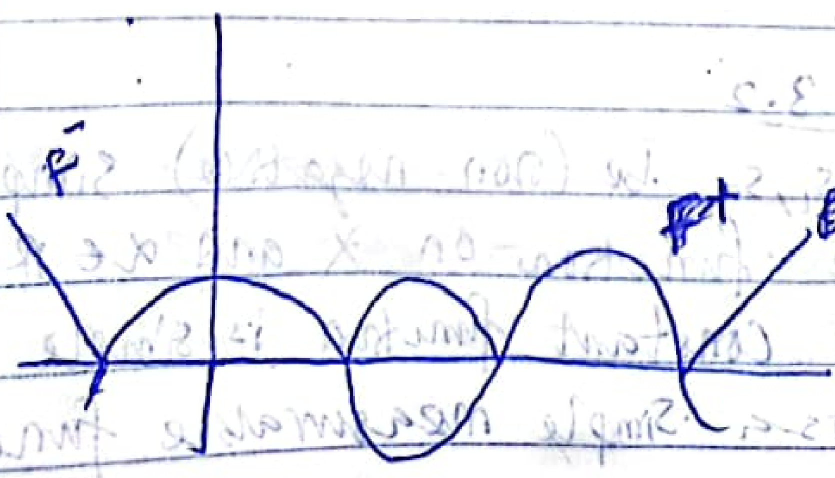
iv. for any measurable set E, $s\chi_E$ is simple measurable.
v. $s_1 \cdot s_2$ is simple measurable.

5/9/2023 <u>Definition</u>
For any function $f : X \to \mathbb{R}^*$ we define:
i. positive part of f by

$$f^+(x) := \begin{cases} f(x), & \text{if } f(x) \geq 0 \\ 0, & \text{if } f(x) < 0 \end{cases}$$

ii. negative part of f by

$$f^-(x) := \begin{cases} -f(x), & \text{if } f(x) \leq 0 \\ 0, & \text{if } f(x) > 0 \end{cases}$$



Remark

i. $f = f^+ - f^-$

ii. $f^+(x) \geq 0$ and $f^-(x) \geq 0 \ \forall x \in X$.

iii. $|f| = f^+ + f^-$

iv. $f^+(x) := \max\{f(x), 0\}$.

$f^-(x) := \max\{-f(x), 0\}$.

Theorem 3.3

A function $f: X \to \mathbb{R}^*$ is measurable if and only if both f^+ and f^- are measurable.

proof

suppose f is measurable. Then for any $c \in \mathbb{R}$

$$(f^+)^{-1}([c,+\infty]) := \begin{cases} f^{-1}([c,+\infty]), & c \geq 0 \\ f^{-1}([0,+\infty]) & c < 0 \end{cases}$$

so that f^+ is measurable

Also,

$$(f^-)^{-1}([-\infty, c]) = \begin{cases} f^{-1}([0,+\infty]), & c > 0 \\ f^{-1}([-c,+\infty]), & c \leq 0 \end{cases}$$

and again f^- is measurable.

conversely, if f^+ and f^- are measurable, then, for any $c \in \mathbb{R}$,

$$f^{-1}([c,+\infty]) = f^{-1}([c,+\infty] \cap [0,+\infty]) \cup f^{-1}([c,+\infty] \cap [-\infty, 0]$$

$= (f^{+})^{-1}([c, +\infty] \cap [0, +\infty]) \cup (f^{-})^{-1}([c, \infty] \cap [-\infty, 0))$

Which is measurable. Thus, f is measurable $\square$.

Theorem 3.4

A nonnegative function $f: X \to \mathbb{R}^*$ is measurable iff there exists an increasing sequence of simple measurable function $\{s_n\}_{n \geq 1}$ such that for all $x \in X$, $\lim_{n \to \infty} s_n(x) = f(x)$

proof

Suppose $\lim_{n \to \infty} s_n(x) = f(x)$. Then for any $c \in \mathbb{R}$,

$$f^{-1}(c, +\infty] = \{x \mid f(x) > c\}$$
$$= \bigcup_{n=1}^{\infty} \{x \mid s_n(x) > c\}$$
$$= \bigcup_{n=1} s_n^{-1}(c, \infty]$$

Thus, $f^{-1}(c, +\infty]$ is measurable and so f too is measurable.

Conversely, (Exercise).

Corollary 3.5

A function $f: X \to \mathbb{R}^*$ is measurable iff there exists a sequence of simple measurable functions $\{s_n\}_{n \geq 1}$ converging to f.

proof

The function f is measurable iff f^+ and f^- are measurable (by theorem 3.3).

iff $\exists \{s_n\}_{n\geq 1}$ & $\{s'_n\}_{n\geq 1}$ increasing sequence of simple measurable fxns such that $\lim_{n\to\infty} s_n(x) = f^+(x)$ and $\lim_{n\to\infty} s'(x) = f^-(x)$ (by theorem 3.4).

But then, the function

$$\phi_n(x) = s_n(x) - s'_n(x)$$

is both simple and measurable. Also,

$$\lim_{n\to\infty} \phi_n(x) = \lim_{n\to\infty} s_n(x) - \lim_{n\to\infty} s'_n(x)$$

$$= f^+(x) - f^-(x) = f(x) \ \square.$$

Theorem 3.6

If $f, g: X \to \mathbb{R}^+$ are measurable functions and $\alpha \in \mathbb{R}$, then

i. αf is measurable

ii. $f+g$ is measurable

iii. $|f|$ is measurable

iv. $f \cdot \chi_E$ is measurable $\forall E$ measurable

v. $f \cdot g$ is measurable

$$fg(x) = f(x) \cdot g(x).$$

Theorem 3.7

Let $\{f_n\}_{n\geq 1}$ be a sequence of measurable functions $f_n : X \to \mathbb{R}^*$ converging to a function $f : X \to \mathbb{R}^*$, i.e. $\lim_{n\to\infty} f_n(x) = f(x)$. Then f is measurable.

proof

Let $\alpha \in \mathbb{R}$. Then

$$\left(\sup_n f_n\right)^{-1}[-\infty, \alpha] = \{x \in X \mid \sup_n f_n(x) \leq \alpha\}$$

$$= \bigcap_{n=1}^{\infty} \{x \in X \mid f_n(x) \leq \infty\}.$$

and

$$\left(\inf_n f_n\right)^{-1}[\alpha, +\infty] = \{x \in X \mid \inf_n f_n(x) \geq \alpha\}$$

$$= \bigcap_{n=1}^{\infty} \{x \in X \mid f_n(x) \geq \alpha\}.$$

since each f_n is measurable, each set $\{x \in X \mid f_n(x) \geq \alpha\}$ and $\{x \in X \mid f_n(x) \leq \alpha\}$ is measurable. Thus, $\sup_n f_n$ and $\inf_n f_n$ are measurable. Therefore

$$\lim_{n\to\infty} \sup f_n(x) = \inf_{n\geq 1} \left\{ \sup_{n\geq m} f_m(x) \right\}$$

and

$$\lim_{n\to\infty} \inf f_n(x) = \sup_{n\geq 1} \left\{ \inf_{n\geq m} f_m(x) \right\}$$

are measurable. Hence,

$$f(x) = \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \inf f_n(x) = \lim_{n \to \infty} \sup f_n(x)$$

is measurable.

<u>Definition</u>

Let (X, Σ, μ) be a measure space. A property P about points $x \in X$ is said hold <u>almost every where</u> with respect to μ if the set $E = \{x \in X \mid P \text{ does not hold at } x\} \in \Sigma$ and $\mu(E) = 0$.

<u>Lemma 3.8.</u>

Let $f, g : X \to \mathbb{R}^*$ such that $f = g$ (a.e) (i.e $\mu(\{x \in X \mid f(x) \neq g(x)\}) = 0$. Then f is measurable iff g is measurable

<u>Definition.</u>

A fn $f : \mathbb{R} \to \mathbb{R}^*$ is Lebesgue measurable if $f^{-1}(I) \in \mathcal{L}_{\mathbb{R}}$ $\forall$ interval $I \subseteq \mathbb{R}^*$

$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \inf f_n(x) = \lim_{n\to\infty} \sup f_n(x) = f(x)$

is measurable.

Definition

Let (X, Σ, μ) be a measure space. A property P about points $x \in X$ is said to hold almost everywhere with respect to μ if the set $E = \{x \in X \mid P \text{ does not hold at } x\} \in \Sigma$ and $\mu(E) = 0$.

Lemma 3.8:

Let $f, g : X \to \mathbb{R}^*$ such that $f = g$ μ-a.e. i.e. $\mu(\{x \in X \mid f(x) \neq g(x)\}) = 0$. Then f is measurable iff g is measurable.

Definition

A fn $f : \mathbb{R} \to \mathbb{R}^*$ is Lebesgue measurable if $f^{-1}(I) \in \mathcal{L}_{\mathbb{R}}$ $\forall$ intervals $I \subseteq \mathbb{R}^*$.

prove $\int s\,dx = \lim_{n\to\infty} \int s_n\,dx$

proof

Since $0 \le s_n \le s$, we have $\int s_n\,dx \le \int s\,dx$,

Hence; $\limsup_{n\to\infty} \int s_n\,dx \le \int s\,dx$.

Let $0 < c < 1$ be arbitrary and let

$B_n : \{x \in X \mid s_n(x) \geq c\,s(x)\}$.

Then $B_n \in S$ and $B_n \subseteq B_{n+1}$ $\forall n$

with $\bigcup_{n=1}^{\infty} B_n = X$ thus we have

$$c\int s(x)\,dx\,(x) = \lim_{n\to\infty} \int_{B_n} c\,s(x)\,dx\,(x)$$

$$\le \liminf_{n\to\infty} \int_{B_n} s_n(x)\,dx\,(x)$$

$$\le \lim_{n\to\infty} \int s_n(x)\,dx\,(x)$$

Since this holds $\forall c$ with $0 < c < 1$ we have

$\int s(x)\,dx\,(x) \le \liminf_{n\to\infty} \int s_n(x)\,dx\,(x)$

$\Rightarrow \int s\,dx = \lim_{n\to\infty} \int s_n\,dx$

$$\lim_{n\to\infty}\int g_n\,d\mu \le \lim_{n\to\infty}\int f_n\,d\mu \le \int f\,d\mu$$

Hence

$$\int f\,d\mu = \int g\,d\mu = \lim_{n\to\infty}\int g_n\,d\mu \le \lim_{n\to\infty}\int f_n\,d\mu \le \int f\,d\mu$$

Non - Negative measurable fn.

A non-negative fn $f: X \to R^*$ is said to be S-measurable if there exists an increasing sequence of fn $\{s_n\}_{n\geq 1}$ in L^+. $\ni$

$$f(x) = \lim_{n\to\infty} s_n(x) \quad \forall x \in X$$

For a fn $f \in L^+$ we define the integral of f w.r.t μ by

$$\int f(x)\,d\mu(x) = \lim_{n\to\infty}\int s_n(x)\,d\mu(x).$$

$$L_0^+ \subseteq L^+.$$

- A fn $f: X \to R^*$ is said to be simple fn if $\exists\ a_1, a_2, \dots, a_n \in X$ & subsets $A_i \subseteq X$ $\ni$ $A_i \cap A_j = \phi$ $i \neq j$ $\cup A_i = X$

$$s(x) := \sum a_i \chi_{A_i}(x),$$

$\Rightarrow \mu(A \cup B) = \sum_{i=1}^{\infty} \mu(\{x_i\}) + \sum_{i=1}^{\infty} \mu(\{x_i\})$
showing that μ is countably additive since $\mathbb{R}$ is uncountable.

Monotone convergence theorem.
Let $\{f_n\}_{n \geq 1}$ be an increasing sequence of functions in L^+ and $f(x) = \lim_{n \to \infty} f_n(x)$, then
$\int f \, dx = \lim_{n \to \infty} \int f_n \, dx$.

proof

since $f_n \in L^+$, then $\exists$ a sequence $\{s_j^n\}_{\geq 1}$ of function in L^+ $\ni$ $\{s_j^n\{x\}\}_{n \geq 1}$ increases to $f_n(x)$ $\forall x$.
$g_n(x) = \max\{s_n^1(x), s_n^2(x), \dots, s_n^n(x)\}$
$g_n \in L^+$ and for every $x \in X$ $\{g_n(x)\}_{n \geq 1}$ is an increasing sequence in $\mathbb{R}^n$.
let $g(x) = \lim_{n \to \infty} g_n(x)$. By definition $g \in L^+$
since $\{f_n\}_{n \geq}$ is increasing and $s_n^j \leq f_j \leq f$
$g_n = \max_{i < j < n} \{s_n^j\} \leq f_n \leq f$

Hence $f \leq g$. proving that $f = g \in L^+$.
Also, by definition
$\int g \, dx = \lim_{n \to \infty} \int g_n \, dx$
since for every n $g_n \leq f_n \leq f$ we get

$$\mu(B) = \mu\left(\bigsqcup_{j=1}^{m} \{x_j\}\right) = \sum_{j=1}^{m} \{x_j\}$$

$$\mu(A \cup B) = \mu\left(\bigsqcup_{i=1}^{n} \bigsqcup_{j=1}^{m} (\{x_i\} \cup \{x_j\})\right)$$
$$= \sum_{i=1}^{n} \mu(\{x_i\}) + \sum_{j=1}^{m} \mu(\{x_j\})$$
$$= \mu(A) + \mu(B) = 0 + 1 = 1$$

showing that μ is finitely additive but not countably additive

Q. 2c.

If we use $\mathbb{R}$ instead of $\mathbb{Q}$, then μ is clearly countably additive since $\mathbb{R}$ is uncountable. To do that, let A be finite and $A^c = B$. Since A is finite and $A^c = B$ is complete,

$\Rightarrow \mu(A) = 0,\ \mu(B) = 1$. since $A, B \in A_0(\mathcal{C})$

$\Rightarrow A = \bigsqcup_{i=1}^{n} \{x_i\}$ and $B = \bigsqcup_{j=1}^{m} \{x_j\}$

$$\mu(A \cup B) = \mu\left(\bigsqcup_{i=1}^{n} \bigsqcup_{j=1}^{m} (\{x_i\} \cup \{x_i\})\right)$$
$$= \mu\left(\bigsqcup_{i=1}^{\infty} \{x_i\}\right) \cup \mu\left(\bigsqcup_{j=1}^{\infty} \{x_i\}\right)$$
$$= \sum_{i=1}^{\infty} \mu(\{x_i\}) + \sum_{j=1}^{\infty} \mu(\{x_j\})$$
$$= \mu(A) + \mu(B)$$
$$= 0 + 1 = 1$$

since for each $x \in \mathbb{Q}$, $\{x\} = A_0(c)$ and so $c \subseteq A_0(c)$.

6. Let M be any other algebra of subsets of $\mathbb{Q}$ with $c \subseteq M$. Let $A \in A_0(c)$. Then either A is finite or A^c is finite. If A is finite then $A = \bigcup_{i=1}^{n} \{x_i\}$ where $x_i \in \mathbb{Q}$. But $\{x_i\} \in c \subseteq M$ and so M is an algebra. It follows that $A \in M$. Thus $A(c) \subseteq M$.

If A^c is finite, then $A^c = \bigcup_{i=1}^{n} \{x_i\}$ where $x \in \mathbb{Q}$. But by similar observation as above, we have $A^c \in M$ and so $A \in M$, thus $A(c) \subseteq M$. This prove the result.

Q-26.

We want to show that μ is finitely additive but not countably additive.

Firstly, we show that A is finitely additive. since $\mathbb{Q}$ is countable and $A(c) = \{A \subseteq \mathbb{Q} \mid A \text{ is finite or } A^c \text{ is finite}\}$

now, let A be finite in $A(c)$ and let $A^c = B$. since $A, B \in A(c)$.

$\Rightarrow A = \bigsqcup_{i=1}^{n} \{x_i\}$ and $B = \bigsqcup_{j=1}^{m} \{x_j\}$

$\mu(A) = \mu(\bigsqcup_{j=1}^{n} \{x_i\}) = \sum_{i=1}^{n} \{x_i\}$

Q. 1b.

Given that $I \in \mathcal{J}$ & $I \subseteq \bigcup_{i=1}^{\infty} I_i$ with $I_i \in \mathcal{J}$, we want to show that l is countably subadditive. Since $I \subseteq \bigcup_{i=1}^{\infty} I_i$ where $I_i \cap I_j = \phi$ with $i \neq j$ Then $l(I) = \sum_{i=1}^{\infty} l(I_i)$ showing that l is countably subadditive.

Q1c.

We want to show that l is countably additive. To do that, let $I \in \mathcal{J}$ to be any finite interval such that

$I = \bigcup_{i=1}^{\infty} I_i$ with $I_i \cap I_j = \phi$ when $i \neq j$.

Then, $l(I) = \sum_{i=1}^{\infty} l(I_i)$

showing that l is countably additive.

Q. 2a.

We want to show that $A_0(C) = \{A \subseteq \Omega :$ either A is finite or A^c is finite$\}$.

$A_0(C)$ is an algebra of subsets of Ω. We need to show that

a. $C \subseteq A_0(C)$

b. $A_0(C)$ is the smallest algebra with the property (a) above.

Q.1.a.

We want to show that λ is monotone and finitely additive. To do that, we need to show ~~that~~ pick any two intervals say A and B in j where $A \subseteq B$ and show that $\lambda(A) \leq \lambda(B)$.

Since $A, B \in j$

$\Rightarrow A = (a, b)$ and $B = (c, d)$

$\Rightarrow \lambda(A) = b - a \leq d - c = \lambda(B)$.

$\Rightarrow \lambda$ is monotone.

Next we prove that λ is finitely additive.

Let $A = \bigcup_{i=1}^{n} A_i$ and $B = \bigcup_{i=1}^{n} B_i$

since both $A, B \in j$

For A, we have

$\lambda(A) = \bigcup_{i=1}^{n} A_i = \sum_{i=1}^{n} \lambda(A_i)$

where $A_i \cap A_j = \emptyset$ for $i \neq j$.

For B, we have

$\lambda(B) = \bigcup_{i=1}^{n} B_i = \sum_{i=1}^{n} \lambda(B_i)$

where $B_i \cap B_j \neq \emptyset$ for $i \neq j$

showing that λ is finitely additive.

Defn : Semi-algebra.

Let X be a non-empty set and let $\mathcal{S}$ be a collection of subsets of X. We say ~~that~~ $\mathcal{S}$ is a semi-algebra of subsets of X if it has the following properties

i. $\phi, X \in \mathcal{S}$

ii $A, B \in \mathcal{S} \Rightarrow A \cap B \in \mathcal{S}$.

iii. $A \in \mathcal{S} \Rightarrow A^c = \bigcup_{i=1}^{n} C_i$, where $C_i \in \mathcal{S}$.